Several aspects of frustration-free quantum systems

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Jan. 19, 2026

About me

- Bachelor: Department of Physics, The University of Tokyo
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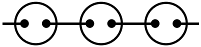
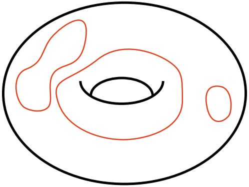
Solvable models:

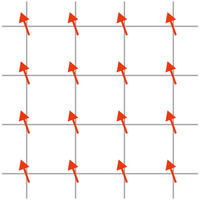
- Free fields, integrable models, conformal field theories
- Frustration-free (FF) systems



Affleck-Kennedy–Lieb–Tasaki model



Toric code

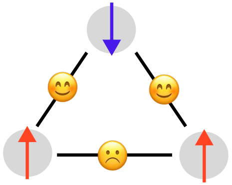


ferromagnetic Heisenberg

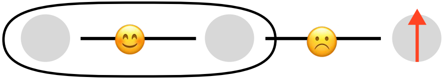
Today's topics

Frustration-freeness as a characterization of quantum systems, rather than an artificial condition for convenience.

What is frustration?



$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \otimes |\uparrow\rangle$$



Definition 1. Frustration-freeness

A Hamiltonian H is called frustration-free (FF) if there exists a decomposition

$$H = \sum_i H_i + \mathrm{const.} \tag{1.1}$$

such that the ground state (GS) minimizes each H_i simultaneously. We can assume $H_i \geq 0$ (positive semidefinite). Then frustration-freeness is equivalent to

$$H_i |\mathrm{GS}\rangle = 0, \quad \forall i. \tag{1.2}$$

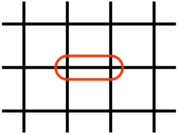
However, this definition is meaningless.

Trivial decomposition: $H = H$.

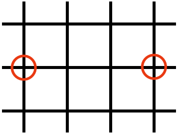
$\rightarrow$ Restrictions must be imposed on the decomposition of H.

Definition 2. k-Locality

We assume each H_i is k-local for a finite k, which means H_i acts non-trivially only on connected k sites.



2-local



4-local

Remark

Determining whether a given state is a GS becomes easier in FF cases (if we already have a nice decomposition).

Examples of FF systems have explicit form of the GS for this reason.

In general, it is computationally hard to determine whether a given Hamiltonian is FF.

- If the decomposition is specified, it is a QMA_1-hard problem.
  Bravyi, arXiv:quant-ph/0602108
- There is a polynomial-time algorithm to search a nice decomposition (with looser restrictions on decomposition than k-locality.)
  Takahashi, Rayudu, Zhou, King, Thompson, Parekh, arXiv:2307.15688

Remark

Non-trivial FF systems need degeneracy of locally favored states.

Let us consider

$$H = H_{12} \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes H_{23}, \tag{1.3}$$

where

$$H_{12} = \mathbb{1} - |\psi_{12}\rangle\langle\psi_{12}|, \quad H_{23} = \mathbb{1} - |\psi_{23}\rangle\langle\psi_{23}|. \tag{1.4}$$

If H is FF under this decomposition,

$$|\mathrm{GS}\rangle = |\psi_{12}\rangle \otimes |\phi_3\rangle = |\phi_1\rangle \otimes |\psi_{23}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle. \tag{1.5}$$

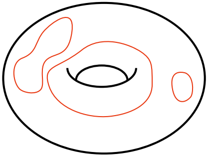
Thus GS must be a trivial tensor product state.

FF-ness is unstable under general perturbations.

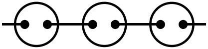
Gapped FF systems vs Gapless FF systems

FF Hamiltonians can approximate general(?) gapped quantum phases.

- Many representative models of gapped phases.



Toric code: $\mathbb{Z}_2$ topological order



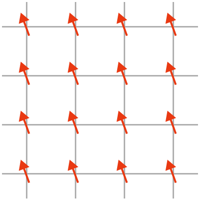
AKLT model: Haldane phase

- In 1D, gapped GS can be approximated by matrix product states.
  Hastings, arXiv:0705.2024
- The GS of a gapped Hamiltonian can be the GS of a quasi-local FF Hamiltonian. Kitaev, Ann. Phys. 321(1), 2-111 (2006)., Sengoku, Watanabe, arXiv:2505.01010

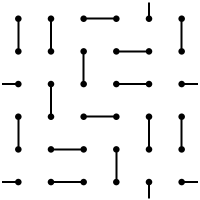
However, gapless FF systems exhibit different low-energy behaviors than typical gapless systems (as we will see).

FF gapless systems are useless as an approximation of gapless systems.

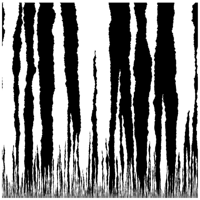
$\leftrightarrow$ FF gapless systems are interesting in their own right.



ferromagnetic Heisenberg



Rokhsar–Kivelson point



critical kinetic Ising

Dynamical exponents

We focus on dynamical exponents.

Definition 3. Spectral gap

Let us take the ground state energy of H to be zero. The spectral gap $\mathrm{gap}(H)$ is the smallest nonzero eigenvalue of H.

Definition 4. Dynamical exponent

For gapless systems, the dynamical exponent z is defined by

$$\mathrm{gap}(H) \sim L^{-z} \tag{2.1}$$

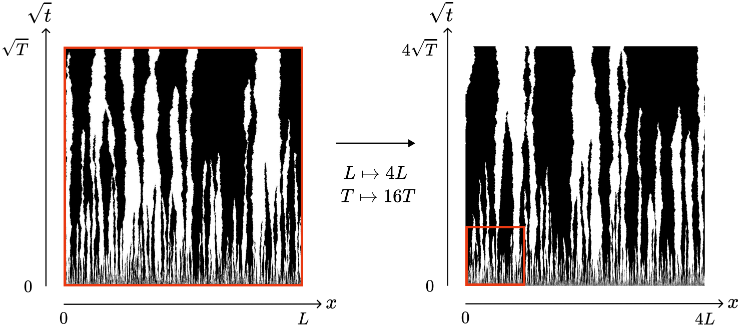
where L is the linear size of the system.

- Typical gapless systems : $z = 1$
- FF gapless systems : $z \geq 2$

Dynamical exponents

Critical points with z are expected to have invariance under the Lifshitz scale transformation given by

$$\boldsymbol{x} \mapsto \lambda \boldsymbol{x}, \quad t \mapsto \lambda^z t, \quad (\lambda > 0). \tag{2.2}$$



Lifshitz scale invariance of the zero-temp. kinetic Ising model ($z = 2$).

Gapless systems with z are expected to have the dispersion relation

$$E_k \sim k^z. \tag{2.3}$$

Conjecture: gapless FF systems have quadratic or softer dispersion.

RM, Soejima, Watanabe, PRB 110, 195140 (2024)

- Coleman's theorem in the contexts of relativistic field theory: Spontaneous symmetry breaking (SSB) of continuous symmetries does not occur in 1+1D systems at $T = 0$.

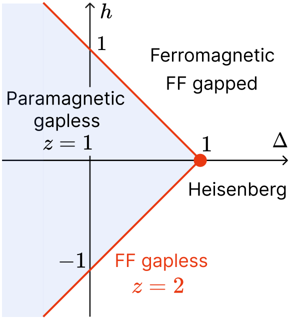
  Coleman, Commun.Math. Phys. 31, 259–264 (1973).
- However, it can occur in 1+1D gapless FF systems because of the quadratic of softer dispersions. Watanabe, Katsura, Lee, PRL 133, 176001 (2024)

Case study: XXZ model + magnetic field

gapless FF $\Rightarrow z \geq 2$

Let us check $z \geq 2$ for gapless FF systems in specific examples.

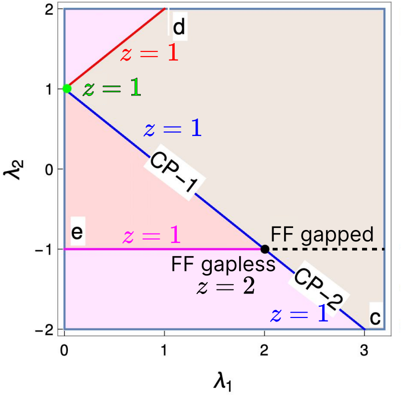
$$H = -\sum_{i=1}^{L}(X_iX_{i+1} + Y_iY_{i+1} + \Delta Z_iZ_{i+1}) + 2h\sum_{i=1}^{L} Z_i + \text{const.}, \tag{2.4}$$



XXZ model with a magnetic field. For example, see the textbook by Franchini (2017).

Case study: quantum Ising model + cluster interaction

$$H = -\sum_{i=1}^{L} (\lambda_1 Z_i Z_{i+1} + \lambda_2 Z_{i-1} X_i Z_{i+1}) + \sum_{i=1}^{L} X_i + \text{const.} \tag{2.5}$$



from Kumar, Kartik, Rahul, Sarkar, Sci. Rep. 11, 1004 (2021). modified

FF systems are fine-tuned, but

- They appear at multicritical points in phase diagrams.
- They appear in the contexts of classical stochastic systems via quantum-classical mapping (as we will see later).

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Rigorous Lower Bound on Dynamical Exponents in Gapless Frustration-Free Systems

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Phys. Rev. X **15**, 041050 – **Published 16 December, 2025**

DOI: https://doi.org/10.1103/d4c4-5p2r

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We show that $z \geq 2$ for a wide range of FF gapless models.

Relating results (limited to the case of open boundary condition)

Gosset, Mozgunov, J. Math. Phys. 57, 091901 (2016). Anshu, PRB 101, 165104 (2020).

Lemm, Xiang, J. Phys. A: Math. Theor. 55 295203 (2022).

The techniques needed for the proof:

Theorem 1. Gosset–Huang inequality

Let H be an FF Hamiltonian and

- G : Projector onto the ground space,
- $\mathcal{O}_{\boldsymbol{x}}, \mathcal{O}'_{\boldsymbol{y}}$: Local operators

Then

$$\frac{|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1}-G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle|}{\|\mathcal{O}_{\boldsymbol{x}}^{\dagger}|\mathrm{GS}\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle\|} \leq 2\exp\left(-C|\boldsymbol{x}-\boldsymbol{y}|\sqrt{\mathrm{gap}(H)}\right), \tag{2.6}$$

where C is a positive constant.

(Gosset and Huang were aware of the application to the gapless FF systems, but they did not demonstrate the scope of its applicability.)

Theorem 2. RM, Soejima, Watanabe PRX 15, 041050 (2025).

FF systems with power-law ground-state correlations satisfy $z \geq 2$.

Proof: Let us assume the system has algebraic correlation functions:

$$|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1} - G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle| \gtrsim \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{\Delta}}, \quad (\Delta > 0) \tag{2.7}$$

From the Gosset–Huang inequality,

$$\frac{1}{L^{\Delta}} \lesssim \frac{|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1} - G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle|}{\|\mathcal{O}^{\dagger}_{\boldsymbol{x}}|\mathrm{GS}\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle\|} \leq 2\exp\left(-CL\sqrt{\mathrm{gap}(H)}\right). \tag{2.8}$$

This inequality breaks for sufficiently large L if $z < 2$. $\square$

c.f. Hastings' inequality for general quantum systems

Hastings, PRL 93, 140402 (2004).

$$\frac{|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1}-G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle|}{\|\mathcal{O}^{\dagger}_{\boldsymbol{x}}|\mathrm{GS}\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle\|} \leq C' \times \exp\left(-C''|\boldsymbol{x}-\boldsymbol{y}|\mathrm{gap}(H)\right). \tag{2.9}$$

The derivation relies on the Lieb-Robinson bound.

This gives a weaker bound $z \geq 1$ for general quantum systems with algebraic correlation functions.

$z \geq 2$ from Gosset–Huang inequality

Our argument is highly general because we do not assume

- boundary condition
- spatial dimension
- structure of the lattice
- translational invariance

Also, our result can be extended to fermionic FF systems.

(Of course, we should explicitly construct an algebraic correlation function.)

Our result: $z \geq 2$ for dynamic critical phenomena

Surprisingly, our framework is also applicable to classical Markov processes, leaving the contexts of quantum systems.

We prove the same bound $z \geq 2$ for dynamic critical phenomena assuming locality and detailed balance.

Critical points	z (numerical)	References
Ising (2D)	$2.1667(5) \geq 2$	Nightingale, Blöte, PRB 62, 1089 (2000).
Ising (3D)	$2.0245(15) \geq 2$	Hasenbusch, PRE 101, 022126 (2020).
Heisenberg (3D)	$2.033(5) \geq 2$	Astillero, Ruiz-Lorenzo, PRE 100, 062117 (2019).
three-state Potts (2D)	$2.193(5) \geq 2$	Murase, Ito, JPSJ 77, 014002 (2008).
four-state Potts (2D)	$2.296(5) \geq 2$	Phys. A: Stat. Mech. Appl. 388, 4379 (2009).

Dynamical exponents of Markov processes relaxing to critical equilibrium states.

Application to Markov processes

We focus on a specific class of FF Hamiltonians.

Definition 5. (Generalized) Rokhsar–Kivelson Hamiltonian

$H^{\mathrm{RK}} = \sum_i H_i^{\mathrm{RK}}$ is a (generalized) RK Hamiltonian if

1. Hamiltonian is FF
2. GS is written as

$$|\Psi_{\mathrm{RK}}\rangle = \sum_{\mathcal{C}} \sqrt{\frac{w(\mathcal{C})}{\mathcal{Z}}}\,|\mathcal{C}\rangle, \quad \mathcal{Z} = \sum_{\mathcal{C}} w(\mathcal{C}), \tag{2.10}$$

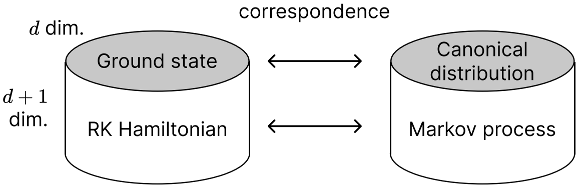
   where $w(\mathcal{C})$ is a Boltzmann weight of a classical statistical system.
3. The off-diagonal elements of H_i are non-positive

There are several names for this class: sto**qu**astic FF Hamiltonian, stochastic matrix form, stochastic quantization.

RK Hamiltonians correspond to Markov processes with local state updates and the detailed balance condition.

Henley, J. Phys.: Condens. Matter 16 S891 (2004).

Castelnovo *et al.*, Ann. Phys. 318, 316 (2005).



Correspondence between RK Hamiltonians and Markov processes.

The correspondence is explicitly given by

$$(W_i)_{\mathcal{C}\mathcal{C}'} := -\sqrt{w(\mathcal{C})}\,(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'}\,\frac{1}{\sqrt{w(\mathcal{C}')}}. \tag{2.11}$$

$W := \sum_i W_i$ is the transition-rate for the corresponding Markov process.

Correspondense between RK Hamiltonians and Markov processes

Imaginary-time Schrödinger eq. $\mathrm{d}\lvert\psi(t)\rangle/\mathrm{d}t = -H^{\mathrm{RK}}\lvert\psi(t)\rangle$	Master eq. $\mathrm{d}p(t)/\mathrm{d}t = W p(t)$
Ground state $\lvert\Psi_{\mathrm{RK}}\rangle = \sum_{\mathcal{C}} \sqrt{w(\mathcal{C})/\mathcal{Z}}\,\lvert\mathcal{C}\rangle$	Steady state $p_{\mathrm{eq}}(\mathcal{C}) = w(\mathcal{C})/\mathcal{Z}$
Symmetricity $(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'} = (H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'}$	Detailed balance condition $(W_i)_{\mathcal{C}\mathcal{C}'} w(\mathcal{C}') = (W_i)_{\mathcal{C}'\mathcal{C}} w(\mathcal{C})$
FF-ness $\langle\Psi_{\mathrm{RK}}\lvert H_i^{\mathrm{RK}} = 0$	Probability conservation $\sum_{\mathcal{C}} (W_i)_{\mathcal{C}\mathcal{C}'} = 0$
Dynamical exponent $\mathrm{gap}(H^{\mathrm{RK}}) \sim L^{-z}$	Dynamical exponent $\tau := 1/\,\mathrm{gap}(-W) \sim L^{z}$

Example: 2+1D kinetic Ising model

- 2+1D kinetic Ising model (heat bath, Gibbs sampling)

Boltzmann weight:

$$w(\mathcal{C}) = \exp\left(\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j\right) \quad (\sigma_i = \pm 1). \tag{2.12}$$

The heat bath (Gibbs sampling) algorithm is given by

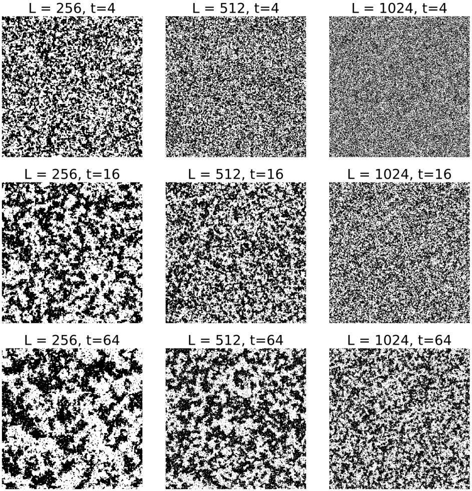
$$(W_i)_{\mathcal{C}'\mathcal{C}} = -(W_i)_{\mathcal{C}\mathcal{C}} = \frac{w(\mathcal{C}')}{w(\mathcal{C}) + w(\mathcal{C}')}, \tag{2.13}$$

where $|\mathcal{C}'\rangle := \sigma_i^x |\mathcal{C}\rangle$. We do not assume any conserved quantity (model A). The corresponding RK Hamiltonian is

$$H_i^{\mathrm{RK}} = \frac{1}{2\cosh(\beta \sum_{j\sim i} Z_j)} \left(\mathrm{e}^{-\beta Z_i \sum_{j\sim i} Z_j} - X_i\right). \tag{2.14}$$

Example: 2+1D kinetic Ising model

At $\beta = \beta_c \approx 0.44$, the relaxation time diverges as $L \to \infty$. ($z \approx 2.17$)



Markov Chain Monte Carlo simulation for 2+1D kinetic Ising model

Critical points	z (numerical)	References
Ising (2D)	$2.1667(5) \geq 2$	Nightingale, Blöte, PRB 62, 1089 (2000).
Ising (3D)	$2.0245(15) \geq 2$	Hasenbusch, PRE 101, 022126 (2020).
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Dynamical exponents of RK Hamiltonians of critical points

RK Hamiltonians of critical points seemed to satisfy $z \geq 2$.

- Conjectured in Isakov, Fendley, Ludwig, Trebst, Troyer, PRB 83, 125114 (2011).
- Previous rigorous result: $z \geq 2 - \eta$. Halperin, PRB 8, 4437 (1973).

Theorem 3. RM, Soejima, Watanabe PRX 15, 041050 (2025).

RK Hamiltonians of critical points satisfy $z \geq 2$.

Our framework: If there is a correlation function such that

$$|\boldsymbol{x}-\boldsymbol{y}| \sim L, \quad \frac{|\langle\Psi|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1}-G)\mathcal{O}'_{\boldsymbol{y}}|\Psi\rangle|}{\|\mathcal{O}^{\dagger}_{\boldsymbol{x}}|\Psi\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\Psi\rangle\|} \gtrsim \frac{1}{L^{\Delta}}, \tag{2.15}$$

then $z \geq 2$. The existence of such a correlation function is quite natural for critical points.

Rigorous discussion (in mathematical sense) for the Ising model is given in RM, Soejima, Watanabe, J Stat Phys 192, 76 (2025).

No-go theorem for local MCMC methods with detailed balance

Rephrasing the theorem in the language of Markov processes, we obtain the following no-go theorem.

No-go theorem

Markov processes for critical points with local state updates and the detailed balance condition satisfy $z \geq 2$.

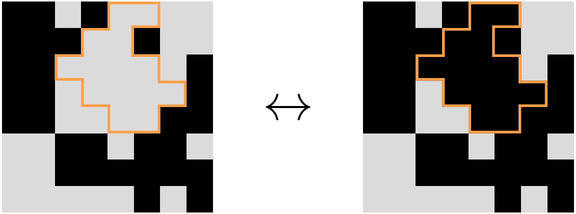
$\rightarrow$ First proof of an empirical fact known in the contexts of dynamic critical phenomena.

Stochastic dynamics with $z < 2$

By violating the assumptions in the no-go theorem, one can create Markov processes with faster relaxation with $z < 2$.

■ Wolff cluster algorithm Wolff, PRL. 62, 361 (1988).

Locality: $\times$,   Detailed balance condition: ✓



State update of the Wolff cluster algorithm

$z \approx 0.3$ for the 2D Ising critical point. Liu *et al.* PRB 89, 054307 (2014).

■ Asymmetric simple exclusion process (ASEP)

Locality: ✓, Detailed balance condition: ×

XXZ model with a non-Hermitian term:

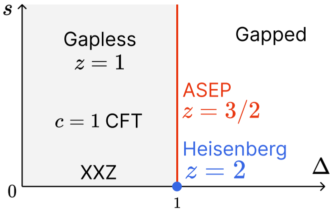
$$H_i = \frac{1}{4}(1 - \Delta Z_i Z_{i+1}) - \frac{1+s}{2}\sigma_i^+ \sigma_{i+1}^- - \frac{1-s}{2}\sigma_i^- \sigma_{i+1}^+ + \frac{s}{2}(Z_i - Z_{i+1}) \quad (2.16)$$

$\Delta < 1$: Gapless phase ($z = 1$)
$\Delta > 1$: Gapped phase
$\Delta = 1$: Stochastic line

- $s = 0$: Heisenberg ($z = 2$, EW class)
- $s > 0$: ASEP ($z = 3/2$, KPZ class)

Kim, PRE 52, 3512 (1995).

Gwa, Spohn, PRA 46, 844 (1992).



Phase diagram of XXZ model with a non-Hermitian term.

This section is based on arXiv:2503.14312 (2025)., arXiv:2503.12879 (2025).

- We established a necessary and sufficient condition for frustration-freeness in free-fermion systems.
- In free-fermion systems, it is clear that why frustration-freeness implies quadratic or softer band dispersions.

$$\hat{H} = \sum_{\boldsymbol{R}\in\Lambda} \hat{H}_{\boldsymbol{R}}, \tag{3.1}$$

$$\begin{aligned}\hat{H}_{\boldsymbol{R}} &= \hat{\boldsymbol{c}}^{\dagger}_{\boldsymbol{R}} H_{\boldsymbol{R}} \hat{\boldsymbol{c}}_{\boldsymbol{R}} + \text{const} \\ &= \sum_{\boldsymbol{\delta},\boldsymbol{\delta}',\sigma,\sigma'} \hat{c}^{\dagger}_{\boldsymbol{R}+\boldsymbol{\delta}\sigma} (H_{\boldsymbol{R}})_{\boldsymbol{R}+\boldsymbol{\delta}\sigma,\boldsymbol{R}+\boldsymbol{\delta}'\sigma'} \hat{c}_{\boldsymbol{R}+\boldsymbol{\delta}'\sigma'} + \text{cosnt}.\end{aligned} \tag{3.2}$$

The constant is chosen so that the GS energy of $\hat{H}_{\boldsymbol{R}}$ is zero. We assumed $\mathrm{U}(1)$ symmetry for simplicity. For more general Hamiltonians including BdG form, see arXiv:2503.12879.

Let us decompose $H_{\boldsymbol{R}}$ into the positive and negative parts as

$$\begin{aligned} H_{\boldsymbol{R}} &= H^{(+)}_{\boldsymbol{R}} + H^{(-)}_{\boldsymbol{R}} \quad (H^{(+)}_{\boldsymbol{R}} \geq 0,\ H^{(-)}_{\boldsymbol{R}} \leq 0) \\ &= \sum_{\alpha=1}^{A_{\boldsymbol{R}}} \mu_{\boldsymbol{R}\alpha} \boldsymbol{\psi}_{\boldsymbol{R}\alpha} \boldsymbol{\psi}^{\dagger}_{\boldsymbol{R}\alpha} - \sum_{\beta=1}^{B_{\boldsymbol{R}}} \nu_{\boldsymbol{R}\beta} \boldsymbol{\phi}_{\boldsymbol{R}\beta} \boldsymbol{\phi}^{\dagger}_{\boldsymbol{R}\beta}, \end{aligned} \tag{3.3}$$

where $\mu_{\boldsymbol{R}\alpha} > 0$ and $-\nu_{\boldsymbol{R}\beta} < 0$ are nonzero eigenvalues of $H_{\boldsymbol{R}}$. $\boldsymbol{\psi}_{\boldsymbol{R}\alpha}$ and $\boldsymbol{\phi}_{\boldsymbol{R}\beta}$ are corresponding orthonormal eigenvectors.

Settings

We define annihilation operators of local orbitals by

$$\hat{\psi}_{\boldsymbol{R}\alpha} := \boldsymbol{\psi}^{\dagger}_{\boldsymbol{R}\alpha}\hat{\boldsymbol{c}}_{\boldsymbol{R}}, \tag{3.4}$$

$$\hat{\phi}_{\boldsymbol{R}\beta} := \boldsymbol{\phi}^{\dagger}_{\boldsymbol{R}\beta}\hat{\boldsymbol{c}}_{\boldsymbol{R}}. \tag{3.5}$$

NOTE: These are not the annihilation operators of eigenmodes of the total Hamiltonian!

Thus, general local terms are rewritten as

$$\begin{aligned}\hat{H}_{\boldsymbol{R}} &= \hat{H}^{(+)}_{\boldsymbol{R}} + \hat{H}^{(-)}_{\boldsymbol{R}} \\ &= \sum_{\alpha=1}^{A_{\boldsymbol{R}}} \mu_{\boldsymbol{R}\alpha}\hat{\psi}^{\dagger}_{\boldsymbol{R}\alpha}\hat{\psi}_{\boldsymbol{R}\alpha} + \sum_{\beta=1}^{B_{\boldsymbol{R}}} \nu_{\boldsymbol{R}\beta}\hat{\phi}_{\boldsymbol{R}\beta}\hat{\phi}^{\dagger}_{\boldsymbol{R}\beta}. \end{aligned} \tag{3.6}$$

FF condition

$$\hat{\psi}_{\boldsymbol{R}\alpha}|\mathrm{GS}\rangle = \hat{\phi}^{\dagger}_{\boldsymbol{R}\beta}|\mathrm{GS}\rangle = 0,\ \forall\alpha,\beta. \tag{3.7}$$

Frustration-free conditions in real space

FF condition

$$\hat{\psi}_{\boldsymbol{R}\alpha}|\mathrm{GS}\rangle = \hat{\phi}^{\dagger}_{\boldsymbol{R}\beta}|\mathrm{GS}\rangle = 0,\ \forall \alpha, \beta. \tag{3.8}$$

Necessary and sufficient condition for frustration-freeness

$$\{\hat{\psi}_{\boldsymbol{R}\alpha}, \hat{\phi}^{\dagger}_{\boldsymbol{R}'\beta}\} = 0 \quad \forall \boldsymbol{R}, \boldsymbol{R}', \alpha, \beta. \tag{3.9}$$

The necessity can be seen by applying $\{\hat{\psi}_{\boldsymbol{R}\alpha}, \hat{\phi}^{\dagger}_{\boldsymbol{R}'\beta}\} \in \mathbb{C}$ to GS:

$$\hat{\psi}_{\boldsymbol{R}\alpha}\hat{\phi}^{\dagger}_{\boldsymbol{R}'\beta}|\mathrm{GS}\rangle + \hat{\phi}^{\dagger}_{\boldsymbol{R}'\beta}\hat{\psi}_{\boldsymbol{R}\alpha}|\mathrm{GS}\rangle = 0. \tag{3.10}$$

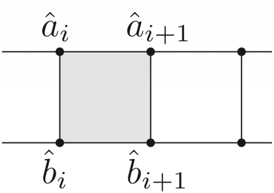
The sufficiency is shown by explicit construction of GS as

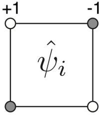
$$|\mathrm{GS}\rangle \propto \prod_{\boldsymbol{R},\beta} \hat{\phi}^{\dagger}_{\boldsymbol{R}\beta}|0\rangle, \tag{3.11}$$

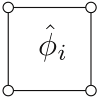
where $|0\rangle$ is the Fock vacuum. If $\{\hat{\phi}_{\boldsymbol{R}\beta}\}$ are linearly dependent, one can choose a linearly independent subset to construct GS.

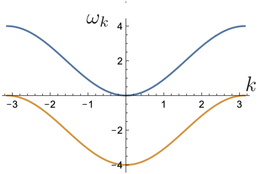
$$\hat{H} = \sum_i \hat{\psi}_i^\dagger \hat{\psi}_i + \sum_i \hat{\phi}_i \hat{\phi}_i^\dagger, \tag{3.12}$$

$$\hat{\psi}_i = \frac{\hat{a}_i - \hat{b}_i - \hat{a}_{i+1} + \hat{b}_{i+1}}{2}, \quad \hat{\phi}_i = \frac{\hat{a}_i + \hat{b}_i + \hat{a}_{i+1} + \hat{b}_{i+1}}{2}. \tag{3.13}$$



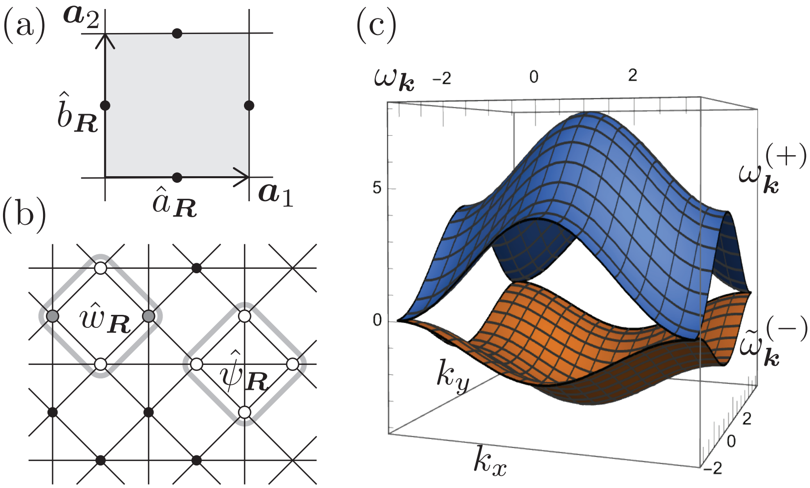






This model satisfies the FF condition:

$$\{\hat{\psi}_i, \hat{\phi}_j^\dagger\} = 0, \ \forall i, j. \tag{3.14}$$



This model hosts spatial conformal invariance arXiv:2511.16496.

An important consequence

An important consequence of the frustration-freeness is that $\hat{H}_{\boldsymbol{R}}^{(+)}$ and $\hat{H}_{\boldsymbol{R}'}^{(-)}$ commute:

$$[\hat{H}_{\boldsymbol{R}}^{(+)}, \hat{H}_{\boldsymbol{R}'}^{(-)}] = \sum_{\alpha=1}^{A_{\boldsymbol{R}}} \sum_{\beta=1}^{B_{\boldsymbol{R}}} \mu_{\boldsymbol{R}\alpha} \nu_{\boldsymbol{R}\beta} \times (\hat{\phi}_{\boldsymbol{R}'\beta}^{\dagger} \{\hat{\phi}_{\boldsymbol{R}'\beta}, \hat{\psi}_{\boldsymbol{R}\alpha}^{\dagger}\} \hat{\psi}_{\boldsymbol{R}\alpha} - \hat{\psi}_{\boldsymbol{R}\alpha}^{\dagger} \{\hat{\psi}_{\boldsymbol{R}\alpha}, \hat{\phi}_{\boldsymbol{R}'\beta}^{\dagger}\} \hat{\phi}_{\boldsymbol{R}'\beta}) = 0. \quad (3.15)$$

This relation implies that the positive and negative parts $\hat{H}^{(\pm)} := \sum_{\boldsymbol{R}} \hat{H}_{\boldsymbol{R}}^{(\pm)}$ independently give positive and negative modes in the energy spectrum.

Reminder:

$$\hat{H}_{\boldsymbol{R}} = \hat{H}_{\boldsymbol{R}}^{(+)} + \hat{H}_{\boldsymbol{R}}^{(-)} = \sum_{\alpha=1}^{A_{\boldsymbol{R}}} \mu_{\boldsymbol{R}\alpha} \hat{\psi}_{\boldsymbol{R}\alpha}^{\dagger} \hat{\psi}_{\boldsymbol{R}\alpha} + \sum_{\beta=1}^{B_{\boldsymbol{R}}} \nu_{\boldsymbol{R}\beta} \hat{\phi}_{\boldsymbol{R}\beta} \hat{\phi}_{\boldsymbol{R}\beta}^{\dagger}.$$

$$\{\hat{\psi}_{\boldsymbol{R}\alpha}, \hat{\phi}_{\boldsymbol{R}'\beta}^{\dagger}\} = 0 \quad \forall \boldsymbol{R}, \boldsymbol{R}', \alpha, \beta.$$

Let us consider translation-invariant cases. Assumption:

$$\hat{H}_{\boldsymbol{R}+\boldsymbol{R}'} = \hat{T}_{\boldsymbol{R}'}\hat{H}_{\boldsymbol{R}}\hat{T}^{\dagger}_{\boldsymbol{R}'}, \tag{3.16}$$

where $\hat{T}_{\boldsymbol{R}}$ is the translation operator. If this is not satisfied, we can always symmetrize the local terms as

$$\hat{H}'_{\boldsymbol{R}} := \frac{1}{V}\sum_{\boldsymbol{R}'}\hat{T}_{\boldsymbol{R}'}\hat{H}_{\boldsymbol{R}-\boldsymbol{R}'}\hat{T}^{\dagger}_{\boldsymbol{R}'}, \tag{3.17}$$

where V is the number of unit cells. (You can easily check that the symmetrized decomposition still satisfies the FF condition.)

Then, we can omit the subscript $\boldsymbol{R}$ in $H_{\boldsymbol{R}}$:

$$H_{\boldsymbol{R}} = \sum_{\alpha=1}^{A}\mu_\alpha\boldsymbol{\psi}_\alpha\boldsymbol{\psi}_\alpha^\dagger - \sum_{\beta=1}^{B}\nu_\beta\boldsymbol{\phi}_\beta\boldsymbol{\phi}_\beta^\dagger. \tag{3.18}$$

$$\hat{H}_{\boldsymbol{R}} = \sum_{\boldsymbol{\delta},\boldsymbol{\delta}',\sigma,\sigma'}\hat{c}^\dagger_{\boldsymbol{R}+\boldsymbol{\delta}\sigma}(H_{\boldsymbol{R}})_{\boldsymbol{\delta}'\sigma,\boldsymbol{\delta}'\sigma'}\hat{c}_{\boldsymbol{R}+\boldsymbol{\delta}''\sigma'} + \mathrm{const.} \tag{3.19}$$

Introducing the Fourier transformation by $\hat{c}^\dagger_{\boldsymbol{R}\sigma} = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{V}} e^{-i\boldsymbol{k}\cdot\boldsymbol{R}} \hat{c}^\dagger_{\boldsymbol{k}\sigma}$, we have

$$\sum_{\boldsymbol{R}} \hat{H}_{\boldsymbol{R}} = \sum_{\boldsymbol{k}} \hat{\boldsymbol{c}}^\dagger_{\boldsymbol{k}} H_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}} + \text{const}, \tag{3.20}$$

$$\sum_{\boldsymbol{R}} \hat{H}^{(+)}_{\boldsymbol{R}} = \sum_{\boldsymbol{k}} \hat{\boldsymbol{c}}^\dagger_{\boldsymbol{k}} H^{(+)}_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}}, \tag{3.21}$$

$$\sum_{\boldsymbol{R}} \hat{H}^{(-)}_{\boldsymbol{R}} = \sum_{\boldsymbol{k}} \hat{\boldsymbol{c}}^\dagger_{\boldsymbol{k}} H^{(-)}_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}} + \text{const}. \tag{3.22}$$

Then, $H^{(+)}_{\boldsymbol{k}}$ and $H^{(-)}_{\boldsymbol{k}}$ give the positive and negative parts of $H_{\boldsymbol{k}}$. These are explicitly given by

$$H^{(+)}_{\boldsymbol{k}} = \sum_{\alpha} \mu_\alpha \boldsymbol{\psi}_\alpha(\boldsymbol{k}) \boldsymbol{\psi}_\alpha(\boldsymbol{k})^\dagger, \quad H^{(-)}_{\boldsymbol{k}} = -\sum_{\beta} \nu_\beta \boldsymbol{\phi}_\beta(\boldsymbol{k}) \boldsymbol{\phi}_\beta(\boldsymbol{k})^\dagger, \tag{3.23}$$

where $\boldsymbol{\psi}_\alpha$ and $\boldsymbol{\phi}_\beta$ are finite-degree vector polynomials in $e^{i\boldsymbol{k}\cdot\boldsymbol{a}_j}$ defined by

$$(\boldsymbol{\psi}_\alpha(\boldsymbol{k}))_\sigma = \sum_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}} (\boldsymbol{\psi}_\alpha)_{\boldsymbol{\delta}\sigma}, \quad (\boldsymbol{\phi}_\beta(\boldsymbol{k}))_\sigma = \sum_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}} (\boldsymbol{\phi}_\beta)_{\boldsymbol{\delta}\sigma}, \tag{3.24}$$

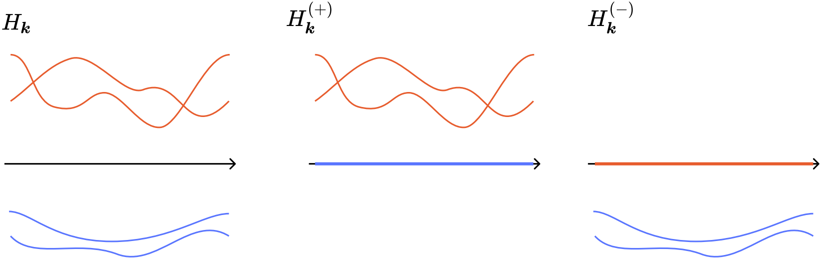
where $\boldsymbol{a}_j$ $(j = 1, \dots, d)$ are primitive lattice vectors.

Frustration-free conditions in momentum space

$$H_{\boldsymbol{k}}^{(+)} = \sum_{\alpha} \mu_\alpha \boldsymbol{\psi}_\alpha(\boldsymbol{k}) \boldsymbol{\psi}_\alpha(\boldsymbol{k})^\dagger, \quad H_{\boldsymbol{k}}^{(-)} = -\sum_{\beta} \nu_\beta \boldsymbol{\phi}_\beta(\boldsymbol{k}) \boldsymbol{\phi}_\beta(\boldsymbol{k})^\dagger.$$

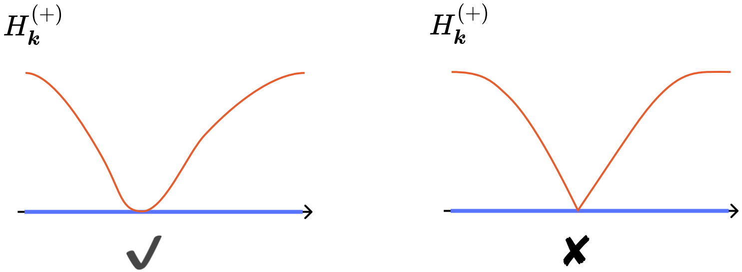
Since $H_{\boldsymbol{k}}^{\pm}$ are Laurent polynomials in $e^{i\boldsymbol{k}\cdot\boldsymbol{a}_j}$ (polynomials in $e^{\pm i\boldsymbol{k}\cdot\boldsymbol{a}_j}$) , both $H_{\boldsymbol{k}}^{(+)}$ and $H_{\boldsymbol{k}}^{(-)}$ form local tight-binding models!

(In general, the positive/negative parts of a given tight-binding Hamiltonian are nonlocal.)



Furthermore, if both $H_{\boldsymbol{k}}^{(+)}$ and $H_{\boldsymbol{k}}^{(-)}$ are nonzero, they must possess flat bands at zero energy.

Now, it is clear why dispersion relations are quadratic or softer in gapless frustration-free free fermions.



Gapless mode only appears when the positive or negative parts touch at zero energy. This touching point is quadratic or softer due to the analyticity of $H_{\boldsymbol{k}}^{(\pm)}$.

Necessary condition for frustration-freeness

$H_{\boldsymbol{k}}^{(\pm)}$ are Laurent polynomials in $e^{i\boldsymbol{k}\cdot\boldsymbol{a}_j}$ where $\boldsymbol{a}_j$ are primitive lattice vectors.

Is this also sufficient?

Frustration-freeness in momentum space

$$H_{\boldsymbol{k}}^{(+)} = \sum_{\alpha} \mu_\alpha \boldsymbol{\psi}_\alpha(\boldsymbol{k})\boldsymbol{\psi}_\alpha(\boldsymbol{k})^\dagger, \quad H_{\boldsymbol{k}}^{(-)} = -\sum_{\beta} \nu_\beta \boldsymbol{\phi}_\beta(\boldsymbol{k})\boldsymbol{\phi}_\beta(\boldsymbol{k})^\dagger.$$

(You can reconstruct $\boldsymbol{\psi}_{\boldsymbol{R}\alpha}$ and $\boldsymbol{\phi}_{\boldsymbol{R}\beta}$ in real space from $\boldsymbol{\psi}_\alpha(\boldsymbol{k})$ and $\boldsymbol{\phi}_\beta(\boldsymbol{k})$.)

$$H_{\boldsymbol{k}}^{(+)} = \sum_{\alpha}[\sqrt{\mu_\alpha}\boldsymbol{\psi}_\alpha(\boldsymbol{k})][\sqrt{\mu_\alpha}\boldsymbol{\psi}_\alpha(\boldsymbol{k})]^\dagger, \quad H_{\boldsymbol{k}}^{(-)} = -\sum_{\beta}[\sqrt{\nu_\beta}\boldsymbol{\phi}_\beta(\boldsymbol{k})][\sqrt{\nu_\beta}\boldsymbol{\phi}_\beta(\boldsymbol{k})]^\dagger.$$

Question: Any operator-valued positive/negative semidefinite Laurent polynomials are decomposed as above?

Let us denote $e^{i\boldsymbol{k}\cdot\boldsymbol{a}_j}$ as z_j.

Practice problem:

$$H_{\boldsymbol{k}}^{(+)} = 4 + z_1 + z_1^* + z_1 z_2^* + z_1^* z_2. \tag{3.25}$$

Answer:

$$\sqrt{\mu_1}\psi_1 = z_1 + 1, \quad \sqrt{\mu_2}\psi_2 = z_1 + z_2. \tag{3.26}$$

$$|z_1 + 1|^2 + |z_1 + z_2|^2 = 4 + z_1 + z_1^* + z_1 z_2^* + z_1^* z_2. \tag{3.27}$$

Frustration-free conditions in momentum space

Question: Any operator-valued positive/negative semidefinite Laurent polynomials are represented as sum of squares?

- Yes, in 1D Rosenblum, J. Math. Anal. Appl., 23, 1 (1963).
- Yes, in 2D Dritschel, Math. Ann. 391, 515–537 (2025).
- No, in 3D or higher Trans. Amer. Math. Soc. 352 (2000).

Necessary and sufficient condition in 1D and 2D

Free fermion Hamiltonian $\hat{H} = \sum_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}}^{\dagger} H_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}}$ is frustration-free if and only if positive/negative parts $H_{\boldsymbol{k}}^{(\pm)}$ of $H_{\boldsymbol{k}}$ are Laurent polynomials in $e^{i\boldsymbol{k}\cdot\boldsymbol{a}_j}$ where $\boldsymbol{a}_j$ are primitive lattice vectors.

Decomposition independent criterion!

Necessary and sufficient condition in 3D and higher

Free fermion Hamiltonian $\hat{H} = \sum_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}}^{\dagger} H_{\boldsymbol{k}} \hat{\boldsymbol{c}}_{\boldsymbol{k}}$ is frustration-free if and only if positive/negative parts $H_{\boldsymbol{k}}^{(\pm)}$ of $H_{\boldsymbol{k}}$ are Laurent polynomials in $e^{i\boldsymbol{k}\cdot\boldsymbol{a}_j}$ where $\boldsymbol{a}_j$ are primitive lattice vectors, and they admit sum of squares decompositions as

$$H_{\boldsymbol{k}}^{(+)} = \sum_{\alpha} \mu_{\alpha} \boldsymbol{\psi}_{\alpha}(\boldsymbol{k}) \boldsymbol{\psi}_{\alpha}(\boldsymbol{k})^{\dagger}, \quad H_{\boldsymbol{k}}^{(-)} = -\sum_{\beta} \nu_{\beta} \boldsymbol{\phi}_{\beta}(\boldsymbol{k}) \boldsymbol{\phi}_{\beta}(\boldsymbol{k})^{\dagger}.$$

(This needs some computational power to check.)

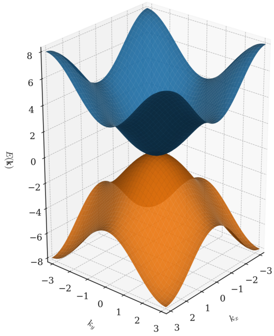
1. Introduction

2. Rigorous lower bound on dynamical exponents

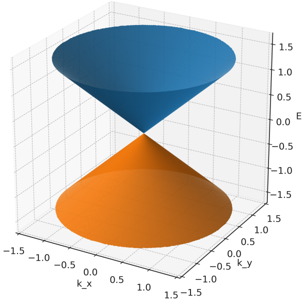
3. Frustration-free free fermions

4. $c = -2$ conformal field theory in quadratic band touching

Quadratic band touching (QBT) in fermion systems provides a distinct low-energy universality class from linear Dirac points.



Quadratic band touching



Dirac cone

non-relativistic $\leftrightarrow$ relativistic

Quadratic band touching

QBT has attracted attention because it is marginally unstable against interactions Sun et al. (2009)., unlike Dirac points.

This instability turns QBT into a platform for studying interaction-driven phases, such as

- nematic order
- quantum anomalous Hall state
- quantum spin Hall state

However, it is important to fully understand non-interacting QBT systems before considering interactions.
I refocus attention on non-interacting QBT as a quantum critical point.

Continuum model

I consider a $(d+1)$-dimensional continuum model of d-component fermions with QBT.

1-form fermions:

$$\hat{\psi}(\boldsymbol{x}) = \hat{\psi}_i(\boldsymbol{x})dx^i, \quad \hat{\psi}^\dagger(\boldsymbol{x}) = \hat{\psi}_i^\dagger(\boldsymbol{x})dx^i, \tag{4.1}$$

The Hamiltonian of the continuum model is given as

$$\begin{aligned}\hat{H} &= t_+(d\hat{\psi}^\dagger, d\hat{\psi}) + t_-(\delta\hat{\psi}, \delta\hat{\psi}^\dagger) \\ &= \int (t_+ d\hat{\psi}^\dagger(\boldsymbol{x}) \wedge \star d\psi(\boldsymbol{x}) + t_- \delta\hat{\psi}(\boldsymbol{x}) \wedge \star\delta\hat{\psi}^\dagger(\boldsymbol{x})),\end{aligned} \tag{4.2}$$

where $t_\pm$ are positive constants.

I will mainly focus on the two-dimensional case $d = 2$ in this talk. In two dimensions, the explicit forms of $d\widehat{\psi}$ and $\delta\widehat{\psi}$ are given as

$$d\widehat{\psi}(\boldsymbol{x}) = (\partial_1 \widehat{\psi}_2(\boldsymbol{x}) - \partial_2 \widehat{\psi}_1(\boldsymbol{x}))dx^1 \wedge dx^2, \tag{4.3}$$

$$\delta\widehat{\psi}(\boldsymbol{x}) = -\partial_1 \widehat{\psi}_1(\boldsymbol{x}) - \partial_2 \widehat{\psi}_2(\boldsymbol{x}), \tag{4.4}$$

and the same applies for $\widehat{\psi}^\dagger$. The Hamiltonian is expressed as

$$\hat{H} = \int \mathrm{d}^2\boldsymbol{x} \begin{pmatrix} \widehat{\psi}_1^\dagger(\boldsymbol{x}) & \widehat{\psi}_2^\dagger(\boldsymbol{x}) \end{pmatrix} H(\nabla) \begin{pmatrix} \widehat{\psi}_1(\boldsymbol{x}) \\ \widehat{\psi}_2(\boldsymbol{x}) \end{pmatrix}, \tag{4.5}$$

$$H(\nabla) = \begin{pmatrix} -t_+\partial_2^2 + t_-\partial_1^2 & -(t_+ + t_-)\partial_1\partial_2 \\ -(t_+ + t_-)\partial_2\partial_1 & -t_+\partial_1^2 + t_-\partial_2^2 \end{pmatrix}. \tag{4.6}$$

In momentum space, the Hamiltonian is expressed as

$$\hat{H} = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \begin{pmatrix} \hat{\psi}^\dagger_{1,\boldsymbol{k}} & \hat{\psi}^\dagger_{2,\boldsymbol{k}} \end{pmatrix} H(\boldsymbol{k}) \begin{pmatrix} \hat{\psi}_{1,\boldsymbol{k}} \\ \hat{\psi}_{2,\boldsymbol{k}} \end{pmatrix}, \tag{4.7}$$

$$\begin{aligned} H(\boldsymbol{k}) &= \begin{pmatrix} t_+ k_2^2 - t_- k_1^2 & (t_+ + t_-)k_1 k_2 \\ (t_+ + t_-)k_2 k_1 & t_+ k_1^2 - t_- k_2^2 \end{pmatrix} \\ &= \frac{t_+ - t_-}{2}(k_1^2 + k_2^2)\sigma_0 - \frac{t_+ + t_-}{2}(k_1^2 - k_2^2)\sigma_z \\ &\quad + (t_+ + t_-)k_1 k_2 \sigma_x. \end{aligned} \tag{4.8}$$

When $d = 2$, this gives a general effective Hamiltonian of rotationally symmetric two-bands system exhibiting QBT (up to unitary transformations).

By diagonalizing this Hamiltonian, the energy dispersions $\epsilon_\pm$ and the Bloch states $b_\pm$ are given as

$$\epsilon_+(\boldsymbol{k}) = t_+|\boldsymbol{k}|^2, \quad \vec{b}_+(\boldsymbol{k}) = \frac{\boldsymbol{k}^\perp}{|\boldsymbol{k}|}, \tag{4.9}$$

$$\epsilon_-(\boldsymbol{k}) = -t_-|\boldsymbol{k}|^2, \quad \vec{b}_-(\boldsymbol{k}) = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \tag{4.10}$$

where $\boldsymbol{k}^\perp = (-k_2, k_1)$. The two bands touch quadratically at $\boldsymbol{k} = \boldsymbol{0}$.

The ground state with all negative-energy states occupied is expressed as

$$|\mathrm{GS}\rangle = \frac{1}{\sqrt{Z}} \prod_{\boldsymbol{k}\neq\boldsymbol{0}} \mathrm{i} g k^j \hat{\psi}^\dagger_{j,\boldsymbol{k}} |0\rangle, \tag{4.11}$$

where $Z = \prod_{\boldsymbol{k}\neq\boldsymbol{0}}(g^2|\boldsymbol{k}|^2)$ and $g = 1/\sqrt{4\pi}$. These factors are introduced for later convenience.

Lattice model

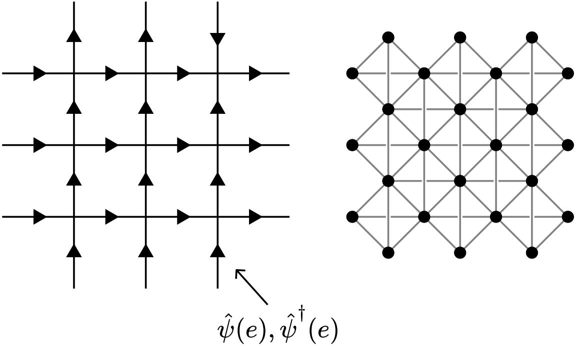


Figure 1: square lattice $\leftrightarrow$ checkerboard lattice

I assign fermions to all edges of the lattice and denote their creation and annihilation operators as $\hat{\psi}^\dagger(e)$ and $\hat{\psi}(e)$, respectively. These satisfy

$$\{\hat{\psi}(e), \hat{\psi}^\dagger(e')\} = \delta_{e,e'}. \tag{4.12}$$

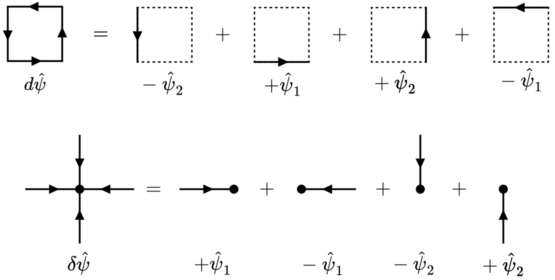
Continuum QBT model:

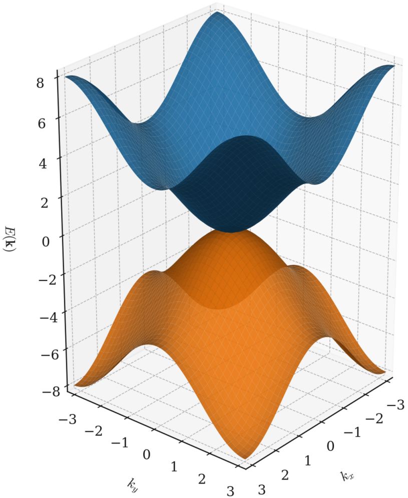
$$\hat{H} = \int d^d x \, (t_+ d\hat{\psi}^\dagger(\boldsymbol{x}) \wedge \star d\hat{\psi}(\boldsymbol{x}) + t\_\delta\hat{\psi}(\boldsymbol{x}) \wedge \star\delta\hat{\psi}^\dagger(\boldsymbol{x})). \tag{4.13}$$

Lattice QBT model:

$$\hat{H} = t_+ \sum_{v \in V} d\hat{\psi}^\dagger(v) d\hat{\psi}(v) + t\_ \sum_{f \in F} \delta\hat{\psi}(f)\delta\hat{\psi}^\dagger(f). \tag{4.14}$$

V: set of vertices, F: set of faces.




8
6
4
2
0
−2
−4
−6
−8
$E(\mathbf{k})$
−3
−2
−1
0
1
2
3
k_y
−3
−2
−1
0
1
2
3
k_x

An important property of this model (for both lattice and continuum) is frustration-freeness, which means the ground state minimizes each term of the Hamiltonian simultaneously.

In the present model, this means

$$d\widehat{\psi}^\dagger(v)d\widehat{\psi}(v)|\mathrm{GS}\rangle = 0, \quad \forall v \in V, \tag{4.15}$$

$$\delta\widehat{\psi}(f)\delta\widehat{\psi}^\dagger(f)|\mathrm{GS}\rangle = 0, \quad \forall f \in F. \tag{4.16}$$

Another expression:

$$d\widehat{\psi}(v)|\mathrm{GS}\rangle = 0, \quad \forall v \in V, \tag{4.17}$$

$$\delta\widehat{\psi}^\dagger(f)|\mathrm{GS}\rangle = 0, \quad \forall f \in F. \tag{4.18}$$

What is missing?

- One-particle energy dispersions and Bloch states are easy, but still many-body ground-states have room for non-trivial physics.
- I discover that the ground states of QBT systems exhibit spatial conformal invariance.

Conformal symmetry

Conformal transformations:

$$x^\mu \mapsto x'^\mu, \quad g_{\mu\nu}(x) \mapsto \Omega(x) g_{\mu\nu}(x) \tag{4.19}$$

Locally, it looks like a scale transformation.

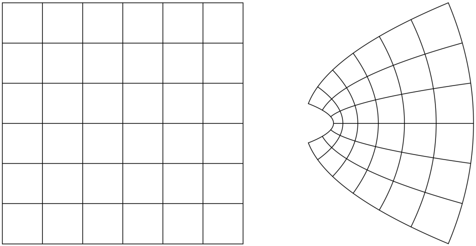


Figure 2: An example of conformal transformation. Angles are preserved, but lengths are not.

Two distinct classes of quantum critical points with conformal symmetry:

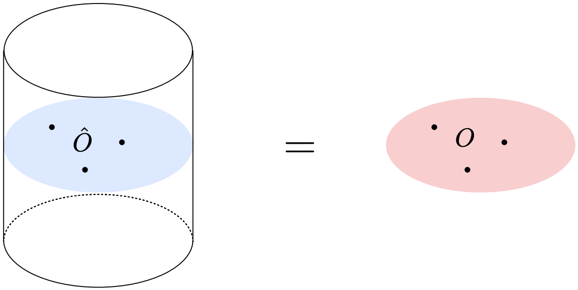
Conformal field theories (as quantum critical points)

- $d+1$-dim. systems with $d+1$-dim. conformal symmetry
- Widely observed.

Conformal quantum critical points (CQCP)

- $d+1$-dim. system
- Non-relativistic $\Rightarrow$ No $d+1$-dim. conformal symmetry
- Ground states exhibits d-dim. spatial conformal symmetry
- Less common and fine-tuned. Often appear as multicritical points.

Spatial conformal symmetry in CQCPs is formulated via the quantum-classical correspondence:



$$\langle \mathrm{GS}|\hat{O}(t=0)|\mathrm{GS}\rangle_{\mathrm{CQCP}_{d+1}} = \langle O\rangle_{\mathrm{CFT}_d} \tag{4.20}$$

Rough explanation of spatial conformal symmetry from frustration-freeness

Lifshitz scale invariance at nonrelativistic quantum critical points:

$$zT_0^0 + T_i^i = \partial_\mu V^\mu, \tag{4.21}$$

where $T^\mu{}_\nu$ is the energy-momentum tensor. We also assume

$$zT_0^0 + T_i^i = 0 \tag{4.22}$$

after improving $T^\mu{}_\nu$. Frustration-freeness implies that

$$T_0^0|\mathrm{GS}\rangle = 0 \Rightarrow T_i^i|\mathrm{GS}\rangle = 0. \tag{4.23}$$

Thus,

$$\langle \mathrm{GS}|T_i^i|\mathrm{GS}\rangle = 0 \tag{4.24}$$

which implies spatial conformal invariance.

$$|\mathrm{GS}\rangle = \frac{1}{\sqrt{Z}} \prod_{\boldsymbol{k}\neq\boldsymbol{0}} \mathrm{i}gk^j \hat{\psi}^\dagger_{j,\boldsymbol{k}}|0\rangle, \tag{4.25}$$

where $Z = \prod_{\boldsymbol{k}\neq\boldsymbol{0}}(g^2|\boldsymbol{k}|^2)$ and $g = 1/\sqrt{4\pi}$.

Let us represent this ground state using a fermionic path integral. For each non-zero mode, I insert the identity

$$x = \int \exp(x\theta_{\boldsymbol{k}})\overleftarrow{\mathrm{d}}\theta_{\boldsymbol{k}}. \tag{4.26}$$

Here, I use right integration $\overleftarrow{\mathrm{d}}\theta_{\boldsymbol{k}} := \overleftarrow{\partial}/\partial\theta_{\boldsymbol{k}}$ to avoid later sign complications. Then, the ground state is expressed as

$$\begin{aligned}
|\mathrm{GS}\rangle &= \frac{1}{\sqrt{Z}} \prod_{\boldsymbol{k}\neq\boldsymbol{0}} \left[\int \exp\left(\mathrm{i}gk^j\hat{\psi}^\dagger_{j,\boldsymbol{k}}\theta_{\boldsymbol{k}}\right)\overleftarrow{\mathrm{d}}\theta_{\boldsymbol{k}}\right]|0\rangle \\
&= \frac{1}{\sqrt{Z}} \int \theta_{\boldsymbol{k}=\boldsymbol{0}} \exp\left(-g\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\mathrm{i}k^j\theta_{\boldsymbol{k}}\hat{\psi}^\dagger_{j,\boldsymbol{k}}\right)|0\rangle\overleftarrow{\mathcal{D}}\theta \\
&= \frac{1}{\sqrt{Z}} \int \theta_{\boldsymbol{k}=\boldsymbol{0}} \exp\left(-g\int \mathrm{d}^2\boldsymbol{x}\,\partial^j\theta(\boldsymbol{x})\hat{\psi}^\dagger_j(\boldsymbol{x})\right)|0\rangle\overleftarrow{\mathcal{D}}\theta.
\end{aligned} \tag{4.27}$$

Thus, the ground state can be represented as

$$|\xi\rangle := \frac{1}{\sqrt{Z}} \int \xi |gd\theta\rangle \overleftarrow{\mathcal{D}}\theta, \tag{4.28}$$

where $\xi := \theta_{\boldsymbol{k}=\boldsymbol{0}}$ is the zero mode and $|gd\theta\rangle$ is a fermionic coherent state given by

$$|gd\theta\rangle := \exp\left(-g \int \mathrm{d}^d\boldsymbol{x}\, \partial^j\theta(\boldsymbol{x})\hat{\psi}_j^\dagger(\boldsymbol{x})\right)|0\rangle. \tag{4.29}$$

This coherent state satisfies

$$\hat{\psi}_i(\boldsymbol{x})|gd\theta\rangle = g\partial_i\theta(\boldsymbol{x})|gd\theta\rangle = \frac{\partial_i\theta(\boldsymbol{x})}{\sqrt{4\pi}}|gd\theta\rangle \tag{4.30}$$

Other degenerate ground states can be constructed by acting the zero-mode creation operators $\hat{\psi}^\dagger_{i,\boldsymbol{k}=0}$ on $|\mathrm{GS}\rangle$.

The bra of the ground state in Eq. (4.28) is given as

$$\langle \xi^* | = \frac{1}{\sqrt{Z}} \int \mathcal{D}\theta^* \langle g d\theta^* | \xi^* . \tag{4.31}$$

Here, θ^* are not the complex conjugates of θ, but independent fields. Then, the norm of the ground state is

$$\begin{aligned} \langle \xi^* | \xi \rangle &= \frac{1}{Z} \int \mathcal{D}\theta^* \langle d\theta^* | \xi^* \int \xi | g d\theta \rangle \overleftarrow{\mathcal{D}}\theta \\ &= \frac{1}{Z} \int \mathcal{D}\theta^* \xi^* \xi \exp\left(g^2 (d\theta^*, d\theta) \right) \overleftarrow{\mathcal{D}}\theta \\ &= \frac{1}{Z} \int \mathcal{D}\theta \mathcal{D}\theta^* \xi^* \xi \exp(-S[\theta, \theta^*]). \end{aligned} \tag{4.32}$$

The normalization constant Z can be regarded as a partition function. The action $S[\theta, \theta^*]$ is given as

$$S[\theta, \theta^*] = \frac{1}{4\pi} \int d^d x \partial_i \theta(\boldsymbol{x}) \partial^i \theta^*(\boldsymbol{x}), \tag{4.33}$$

which coincides with that of the symplectic fermion theory.

The correlation functions in the QBT systems correspond exactly to those of symplectic fermion. For the two-point function, we have

$$\begin{aligned} \langle \xi^* | \hat{\psi}_i^\dagger(\boldsymbol{x}) \hat{\psi}_j(\boldsymbol{y}) | \xi \rangle &= \frac{1}{Z} \int \mathcal{D}\theta^* \xi^* \langle g d\theta^* | \hat{\psi}_i^\dagger(\boldsymbol{x}) \hat{\psi}_j(\boldsymbol{y}) \int \xi | g d\theta \rangle \overleftarrow{\mathcal{D}}\theta \\ &= \frac{g^2}{Z} \int \mathcal{D}\theta \mathcal{D}\theta^* \xi^* \partial_i \theta^*(\boldsymbol{x}) \partial_j \theta(\boldsymbol{y}) \xi e^{-S[\theta,\theta^*]} \\ &= \frac{1}{4\pi} \langle \xi^* \partial_i \theta^*(\boldsymbol{x}) \partial_j \theta(\boldsymbol{y}) \xi \rangle, \end{aligned} \tag{4.34}$$

where we have defined

$$\langle X \rangle := \frac{1}{Z} \int \mathcal{D}\theta \mathcal{D}\theta^* X e^{-S[\theta,\theta^*]}. \tag{4.35}$$

For general correlation functions, we have

$$\langle \xi^* | F[\hat{\psi}^\dagger] G[\hat{\psi}] | \xi \rangle = \langle \xi^* F[g d\theta^*] G[g d\theta] \xi \rangle, \tag{4.36}$$

for arbitrary functionals F and G. This correspondence is summarized as

$$\hat{\psi} \leftrightarrow \frac{d\theta}{\sqrt{4\pi}}, \quad \hat{\psi}^\dagger \leftrightarrow \frac{d\theta^*}{\sqrt{4\pi}}. \tag{4.37}$$

Note that in addition to simply making this replacement, we need to additionally insert zero modes $\xi^* \xi$.

Additional results

For more details, please refer to my paper arxiv:2511.16496.

- There exist anyon-like excitations in (2+1)D QBT systems originating from the underlying symplectic fermion.
- Moving excitations along non-contractible loops induces transitions between topologically degenerate ground states.
- Action of 2π rotation for these anyons exhibit a Jordan block structure.