$c = -2$ conformal field theory in quadratic band touching

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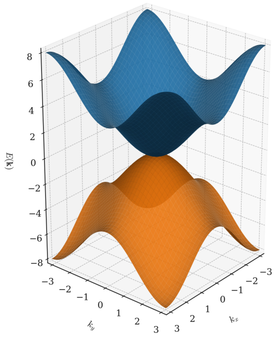
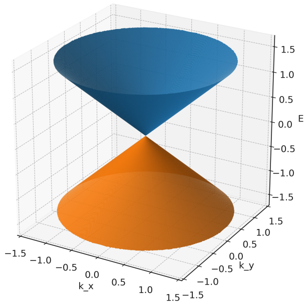
arxiv:2511.xxxxx (coming soon)

Quadratic band touching (QBT) in fermion systems provides a distinct low-energy universality class from linear Dirac points.



Quadratic band touching



Dirac cone

non-relativistic ↔ relativistic

Quadratic band touching

QBT has attracted attention because it is marginally unstable against interactions Sun et al. (2009)., unlike Dirac points.

This instability turns QBT into a platform for studying interaction-driven phases, such as

- nematic order
- quantum anomalous Hall state
- quantum spin Hall state

However, it is important to fully understand non-interacting QBT systems before considering interactions.
I refocus attention on non-interacting QBT as a quantum critical point.

What is missing?

QBT model in momentum space:

$$H(\boldsymbol{k}) = \begin{pmatrix} t_+ k_2^2 - t_- k_1^2 & (t_+ + t_-)k_1 k_2 \\ (t_+ + t_-)k_2 k_1 & t_+ k_1^2 - t_- k_2^2 \end{pmatrix} \tag{1.1}$$

Quite easy to solve:

$$\epsilon_+(\boldsymbol{k}) = t_+|\boldsymbol{k}|^2, \quad \vec{b}_+(\boldsymbol{k}) = \frac{1}{|\boldsymbol{k}|}\begin{pmatrix} -k_2 \\ k_1 \end{pmatrix}, \epsilon_-(\boldsymbol{k}) = -t_-|\boldsymbol{k}|^2, \quad \vec{b}_-(\boldsymbol{k}) = \frac{1}{|\boldsymbol{k}|}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \tag{1.2}$$

Are these all about this system?

Actually, studies have overlooked an essential aspect of non-interacting QBT systems!

What is missing?

- One-particle energy dispersions and Bloch states are easy, but still many-body ground-states have room for non-trivial physics.
- I discover that the ground states of QBT systems exhibit spatial conformal invariance.

Conformal transformations:

$$x^\mu \mapsto x'^\mu, \quad g_{\mu\nu}(x) \mapsto \Omega(x) g_{\mu\nu}(x) \tag{1.3}$$

Locally, it looks like a scale transformation.

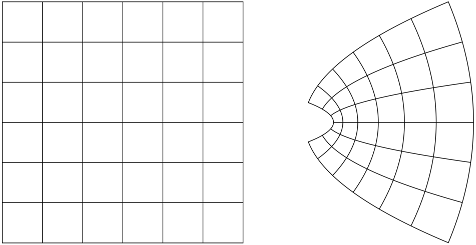


Figure 1: An example of conformal transformation. Angles are preserved, but lengths are not.

Conformal quantum critical points (CQCP)

Two distinct classes of quantum critical points with conformal symmetry:

Conformal field theories (as quantum critical points)

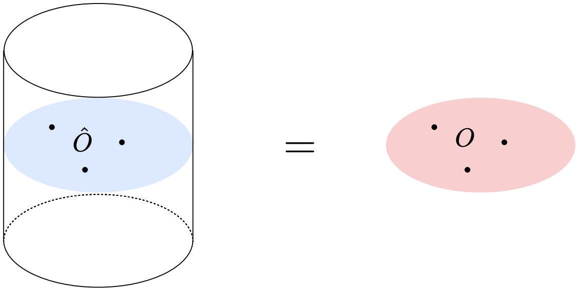
- $d+1$-dim. systems with $d+1$-dim. conformal symmetry
- Widely observed.

Conformal quantum critical points (CQCP) $\leftarrow$ Today's focus

- $d+1$-dim. system
- Non-relativistic $\Rightarrow$ No $d+1$-dim. conformal symmetry
- Ground states exhibits d-dim. spatial conformal symmetry
- Less common and fine-tuned. Often appear as multicritical points.

Conformal quantum critical points (CQCP)

Spatial conformal symmetry in CQCPs is formulated via the quantum-classical correspondence:



$$\langle \mathrm{GS}|\hat{O}(t=0)|\mathrm{GS}\rangle_{\mathrm{CQCP}_{d+1}} = \langle O\rangle_{\mathrm{CFT}_d} \tag{1.4}$$

Conformal quantum critical points (CQCP)

RK state Rokhsar Kivelson (1988). Henley (2004). Castelnovo et al. (2005). :

$$|\mathrm{GS}\rangle := \frac{1}{\sqrt{Z}} \sum_C \sqrt{e^{-\beta E(C)}}\, |C\rangle, \quad Z := \sum_C e^{-\beta E(C)}. \tag{1.5}$$

- C : Classical configurations (e.g. spin config. in the Ising model)
- $\langle C|C'\rangle = \delta_{CC'}$

Quantum-classical correspondence:

$$\langle F(C)\rangle := \frac{1}{Z} \sum_{C\in\mathcal{C}} F(C) e^{-\beta E(C)} = \langle \mathrm{GS}|\hat{F}|\mathrm{GS}\rangle, \quad \hat{F} = \sum_{C\in\mathcal{C}} F(C)|C\rangle\langle C| \tag{1.6}$$

Parent Hamiltonians of RK states are CQCP if the corresponding classical model is critical.

(However, it cannot be applied to fermionic systems.)

Main results

Let us turn back to the QBT systems. In this talk, I present the following results:

- $d+1$-dim. QBT systems form a CQCP corresponding d-dim. symplectic fermion theory.
- The explicit quantum-classical correspondence is written down.
- There exist anyon-like excitations in (2+1)D QBT systems originating from the underlying symplectic fermion.
- Moving anyons along non-contractible loops induces transitions between topologically degenerate ground states.
- Action of 2π rotation for these anyons exhibit a Jordan block structure.

I consider a $(d+1)$-dimensional continuum model of d-component fermions with QBT.

1-form fermions:

$$\widehat{\psi}(\boldsymbol{x}) = \widehat{\psi}_i(\boldsymbol{x})dx^i, \quad \widehat{\psi}^\dagger(\boldsymbol{x}) = \widehat{\psi}_i^\dagger(\boldsymbol{x})dx^i, \tag{2.1}$$

The Hamiltonian of the continuum model is given as

$$\begin{aligned} \hat{H} &= t_+(d\widehat{\psi}^\dagger, d\widehat{\psi}) + t_-(\delta\widehat{\psi}, \delta\widehat{\psi}^\dagger) \\ &= \int (t_+ d\widehat{\psi}^\dagger(\boldsymbol{x}) \wedge \star d\psi(\boldsymbol{x}) + t_- \delta\widehat{\psi}(\boldsymbol{x}) \wedge \star\delta\widehat{\psi}^\dagger(\boldsymbol{x})), \end{aligned} \tag{2.2}$$

where $t_\pm$ are positive constants.

I will mainly focus on the two-dimensional case $d = 2$ in this talk. In two dimensions, the explicit forms of $d\widehat{\psi}$ and $\delta\widehat{\psi}$ are given as

$$d\widehat{\psi}(\boldsymbol{x}) = (\partial_1\widehat{\psi}_2(\boldsymbol{x}) - \partial_2\widehat{\psi}_1(\boldsymbol{x}))dx^1 \wedge dx^2, \tag{2.3}$$

$$\delta\widehat{\psi}(\boldsymbol{x}) = -\partial_1\widehat{\psi}_1(\boldsymbol{x}) - \partial_2\widehat{\psi}_2(\boldsymbol{x}), \tag{2.4}$$

and the same applies for $\widehat{\psi}^\dagger$. The Hamiltonian is expressed as

$$\hat{H} = \int \mathrm{d}^2\boldsymbol{x} \begin{pmatrix} \widehat{\psi}_1^\dagger(\boldsymbol{x}) & \widehat{\psi}_2^\dagger(\boldsymbol{x}) \end{pmatrix} H(\nabla) \begin{pmatrix} \widehat{\psi}_1(\boldsymbol{x}) \\ \widehat{\psi}_2(\boldsymbol{x}) \end{pmatrix}, \tag{2.5}$$

$$H(\nabla) = \begin{pmatrix} -t_+\partial_2^2 + t_-\partial_1^2 & -(t_+ + t_-)\partial_1\partial_2 \\ -(t_+ + t_-)\partial_2\partial_1 & -t_+\partial_1^2 + t_-\partial_2^2 \end{pmatrix}. \tag{2.6}$$

Continuum model

In momentum space, the Hamiltonian is expressed as

$$\hat{H} = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \begin{pmatrix} \hat{\psi}^\dagger_{1,\boldsymbol{k}} & \hat{\psi}^\dagger_{2,\boldsymbol{k}} \end{pmatrix} H(\boldsymbol{k}) \begin{pmatrix} \hat{\psi}_{1,\boldsymbol{k}} \\ \hat{\psi}_{2,\boldsymbol{k}} \end{pmatrix}, \tag{2.7}$$

$$\begin{aligned} H(\boldsymbol{k}) &= \begin{pmatrix} t_+ k_2^2 - t_- k_1^2 & (t_+ + t_-) k_1 k_2 \\ (t_+ + t_-) k_2 k_1 & t_+ k_1^2 - t_- k_2^2 \end{pmatrix} \\ &= \frac{t_+ - t_-}{2}(k_1^2 + k_2^2)\sigma_0 - \frac{t_+ + t_-}{2}(k_1^2 - k_2^2)\sigma_z \\ &\quad + (t_+ + t_-) k_1 k_2 \sigma_x. \end{aligned} \tag{2.8}$$

When $d = 2$, this gives a general effective Hamiltonian of rotationally symmetric two-bands system exhibiting QBT (up to unitary transformations).

By diagonalizing this Hamiltonian, the energy dispersions $\epsilon_\pm$ and the Bloch states $b_\pm$ are given as

$$\epsilon_+(\boldsymbol{k}) = t_+|\boldsymbol{k}|^2, \quad \vec{b}_+(\boldsymbol{k}) = \frac{\boldsymbol{k}^\perp}{|\boldsymbol{k}|}, \tag{2.9}$$

$$\epsilon_-(\boldsymbol{k}) = -t_-|\boldsymbol{k}|^2, \quad \vec{b}_-(\boldsymbol{k}) = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \tag{2.10}$$

where $\boldsymbol{k}^\perp = (-k_2, k_1)$. The two bands touch quadratically at $\boldsymbol{k} = \boldsymbol{0}$.

The ground state with all negative-energy states occupied is expressed as

$$|\mathrm{GS}\rangle = \frac{1}{\sqrt{Z}} \prod_{\boldsymbol{k} \neq \boldsymbol{0}} \mathrm{i} g k^j \hat{\psi}^\dagger_{j,\boldsymbol{k}} |0\rangle, \tag{2.11}$$

where $Z = \prod_{\boldsymbol{k}\neq\boldsymbol{0}}(g^2|\boldsymbol{k}|^2)$ and $g = 1/\sqrt{4\pi}$. These factors are introduced for later convenience.

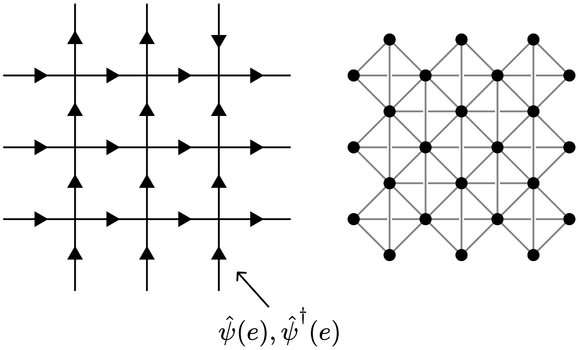


Figure 2: square lattice $\leftrightarrow$ checkerboard lattice

I assign fermions to all edges of the lattice and denote their creation and annihilation operators as $\hat{\psi}^{\dagger}(e)$ and $\hat{\psi}(e)$, respectively. These satisfy

$$\{\hat{\psi}(e), \hat{\psi}^{\dagger}(e')\} = \delta_{e,e'}. \tag{3.1}$$

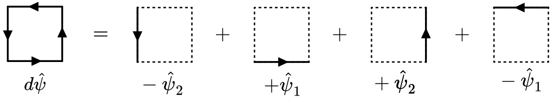
Continuum QBT model:

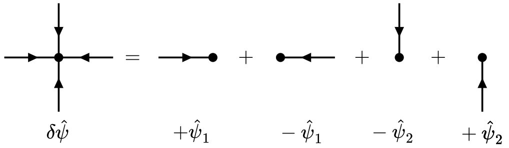
$$\hat{H} = \int d^d x \, (t_+ d\hat{\psi}^\dagger(\boldsymbol{x}) \wedge \star d\hat{\psi}(\boldsymbol{x}) + t_- \delta\hat{\psi}(\boldsymbol{x}) \wedge \star\delta\hat{\psi}^\dagger(\boldsymbol{x})). \tag{3.2}$$

Lattice QBT model:

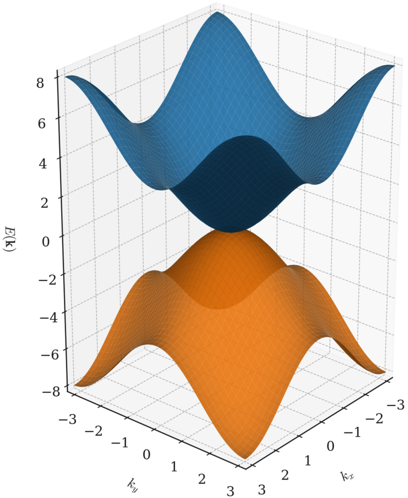
$$\hat{H} = t_+ \sum_{v \in V} d\hat{\psi}^\dagger(v) d\hat{\psi}(v) + t_- \sum_{f \in F} \delta\hat{\psi}(f)\delta\hat{\psi}^\dagger(f). \tag{3.3}$$

V: set of vertices, F: set of faces.





Lattice model



An important property of this model (for both lattice and continuum) is frustration-freeness, which means the ground state minimizes each term of the Hamiltonian simultaneously.

In the present model, this means

$$d\hat{\psi}^\dagger(v)d\hat{\psi}(v)|\mathrm{GS}\rangle = 0, \quad \forall v \in V, \tag{3.4}$$
$$\delta\hat{\psi}(f)\delta\hat{\psi}^\dagger(f)|\mathrm{GS}\rangle = 0, \quad \forall f \in F. \tag{3.5}$$

Another expression:

$$d\hat{\psi}(v)|\mathrm{GS}\rangle = 0, \quad \forall v \in V, \tag{3.6}$$
$$\delta\hat{\psi}^\dagger(f)|\mathrm{GS}\rangle = 0, \quad \forall f \in F. \tag{3.7}$$

Correspondence to symplectic fermion

$$|\mathrm{GS}\rangle = \frac{1}{\sqrt{Z}} \prod_{\boldsymbol{k}\neq\boldsymbol{0}} \mathrm{i}gk^j \hat{\psi}^\dagger_{j,\boldsymbol{k}}|0\rangle, \tag{4.1}$$

where $Z = \prod_{\boldsymbol{k}\neq\boldsymbol{0}}(g^2|\boldsymbol{k}|^2)$ and $g = 1/\sqrt{4\pi}$.

Let us represent this ground state using a fermionic path integral. For each non-zero mode, I insert the identity

$$x = \int \exp(x\theta_{\boldsymbol{k}})\overleftarrow{\mathrm{d}}\theta_{\boldsymbol{k}}. \tag{4.2}$$

Here, I use right integration $\overleftarrow{\mathrm{d}}\theta_{\boldsymbol{k}} := \overleftarrow{\partial}/\partial\theta_{\boldsymbol{k}}$ to avoid later sign complications. Then, the ground state is expressed as

$$\begin{aligned}|\mathrm{GS}\rangle &= \frac{1}{\sqrt{Z}} \prod_{\boldsymbol{k}\neq\boldsymbol{0}} \left[\int \exp\left(\mathrm{i}gk^j\hat{\psi}^\dagger_{j,\boldsymbol{k}}\theta_{\boldsymbol{k}}\right)\overleftarrow{\mathrm{d}}\theta_{\boldsymbol{k}}\right]|0\rangle \\ &= \frac{1}{\sqrt{Z}} \int \theta_{\boldsymbol{k}=\boldsymbol{0}} \exp\left(-g\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\mathrm{i}k^j\theta_{\boldsymbol{k}}\hat{\psi}^\dagger_{j,\boldsymbol{k}}\right)|0\rangle\overleftarrow{\mathcal{D}}\theta \\ &= \frac{1}{\sqrt{Z}} \int \theta_{\boldsymbol{k}=\boldsymbol{0}} \exp\left(-g\int \mathrm{d}^2\boldsymbol{x}\,\partial^j\theta(\boldsymbol{x})\hat{\psi}^\dagger_j(\boldsymbol{x})\right)|0\rangle\overleftarrow{\mathcal{D}}\theta.\end{aligned} \tag{4.3}$$

Thus, the ground state can be represented as

$$|\xi\rangle := \frac{1}{\sqrt{Z}} \int \xi |gd\theta\rangle \overleftarrow{\mathcal{D}}\theta, \tag{4.4}$$

where $\xi := \theta_{\boldsymbol{k}=\mathbf{0}}$ is the zero mode and $|gd\theta\rangle$ is a fermionic coherent state given by

$$|gd\theta\rangle := \exp\left(-g \int \mathrm{d}^d\boldsymbol{x}\, \partial^j\theta(\boldsymbol{x})\hat{\psi}_j^\dagger(\boldsymbol{x})\right)|0\rangle. \tag{4.5}$$

This coherent state satisfies

$$\hat{\psi}_i(\boldsymbol{x})|gd\theta\rangle = g\partial_i\theta(\boldsymbol{x})|gd\theta\rangle = \frac{\partial_i\theta(\boldsymbol{x})}{\sqrt{4\pi}}|gd\theta\rangle \tag{4.6}$$

Other degenerate ground states can be constructed by acting the zero-mode creation operators $\hat{\psi}^\dagger_{i,\boldsymbol{k}=0}$ on $|\mathrm{GS}\rangle$.

Correspondence to symplectic fermion

The bra of the ground state in Eq. (4.4) is given as

$$\langle \xi^* | = \frac{1}{\sqrt{Z}} \int \mathcal{D}\theta^* \langle g d\theta^* | \xi^* . \tag{4.7}$$

Here, θ^* are not the complex conjugates of θ, but independent fields. Then, the norm of the ground state is

$$\begin{aligned} \langle \xi^* | \xi \rangle &= \frac{1}{Z} \int \mathcal{D}\theta^* \langle d\theta^* | \xi^* \int \xi | g d\theta \rangle \overleftarrow{\mathcal{D}}\theta \\ &= \frac{1}{Z} \int \mathcal{D}\theta^* \xi^* \xi \exp\left(g^2 (d\theta^*, d\theta)\right) \overleftarrow{\mathcal{D}}\theta \\ &= \frac{1}{Z} \int \mathcal{D}\theta \mathcal{D}\theta^* \xi^* \xi \exp(-S[\theta, \theta^*]). \end{aligned} \tag{4.8}$$

The normalization constant Z can be regarded as a partition function. The action $S[\theta, \theta^*]$ is given as

$$S[\theta, \theta^*] = \frac{1}{4\pi} \int d^d x \partial_i \theta(\boldsymbol{x}) \partial^i \theta^*(\boldsymbol{x}), \tag{4.9}$$

which coincides with that of the symplectic fermion theory.

Correspondence to symplectic fermion

The correlation functions in the QBT systems correspond exactly to those of symplectic fermion. For the two-point function, we have

$$\begin{aligned}\langle \xi^* | \hat{\psi}_i^\dagger(\boldsymbol{x}) \hat{\psi}_j(\boldsymbol{y}) | \xi \rangle &= \frac{1}{Z} \int \mathcal{D}\theta^* \xi^* \langle g d\theta^* | \hat{\psi}_i^\dagger(\boldsymbol{x}) \hat{\psi}_j(\boldsymbol{y}) \int \xi | g d\theta \rangle \overleftarrow{\mathcal{D}}\theta \\ &= \frac{g^2}{Z} \int \mathcal{D}\theta \mathcal{D}\theta^* \xi^* \partial_i \theta^*(\boldsymbol{x}) \partial_j \theta(\boldsymbol{y}) \xi e^{-S[\theta,\theta^*]} \\ &= \frac{1}{4\pi} \langle \xi^* \partial_i \theta^*(\boldsymbol{x}) \partial_j \theta(\boldsymbol{y}) \xi \rangle, \end{aligned} \tag{4.10}$$

where we have defined

$$\langle X \rangle := \frac{1}{Z} \int \mathcal{D}\theta \mathcal{D}\theta^* X e^{-S[\theta,\theta^*]}. \tag{4.11}$$

Correspondence to symplectic fermion

For general correlation functions, we have

$$\langle \xi^* | F[\hat{\psi}^\dagger] G[\hat{\psi}] | \xi \rangle = \langle \xi^* F[gd\theta^*] G[gd\theta] \xi \rangle, \tag{4.12}$$

for arbitrary functionals F and G. This correspondence is summarized as

$$\hat{\psi} \leftrightarrow \frac{d\theta}{\sqrt{4\pi}}, \quad \hat{\psi}^\dagger \leftrightarrow \frac{d\theta^*}{\sqrt{4\pi}}. \tag{4.13}$$

Note that in addition to simply making this replacement, we need to additionally insert zero modes $\xi^* \xi$.

An aside: Correlation functions

Two-point correlation functions of θ :

$$\langle \xi^* \xi \theta^\alpha(\boldsymbol{x}) \theta^\beta(\boldsymbol{y}) \rangle = -\varepsilon_{\alpha\beta} \ln |\boldsymbol{x} - \boldsymbol{y}|^2, \tag{4.14}$$

where $(\theta^1, \theta^2) = (\theta, \theta^*)$ and $\varepsilon_{12} = -\varepsilon_{21} = 1.$

Multi-point:

$$\langle \xi^* \xi \theta^{\alpha_1}(\boldsymbol{x}_1) \cdots \theta^{\alpha_{2n}}(\boldsymbol{x}_{2n}) \rangle = \mathrm{Pf} \left[-\varepsilon_{\alpha_i \beta_j} \ln |\boldsymbol{x}_i - \boldsymbol{x}_j|^2 \right]_{1 \le i,j \le 2n} \tag{4.15}$$

Two-point correlation of $\hat{\psi}$:

$$\langle \xi^* | \hat{\psi}_i^\dagger(\boldsymbol{x}) \hat{\psi}_j(\boldsymbol{y}) | \xi \rangle = \langle \xi^* \partial_i \theta^*(\boldsymbol{x}) \partial_j \theta(\boldsymbol{y}) \xi \rangle = -\frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \ln |\boldsymbol{x} - \boldsymbol{y}|^2. \tag{4.16}$$

Scale invariance means that fields transform under scaling $\boldsymbol{y} = \lambda \boldsymbol{x}$ as

$$\Phi_\Delta(\boldsymbol{x}) \mapsto \lambda^\Delta \Phi_\Delta(\boldsymbol{y}), \tag{5.1}$$

where Δ is called the scaling dimension.

In ordinary CFTs, a primary field $\phi_h(z)$ with conformal weight h transforms under $z \mapsto w(z)$ as

$$\phi_h(z) \mapsto \left(\frac{dw}{dz}\right)^h \phi_h(w). \tag{5.2}$$

In log CFTs, there exist logarithmic partners $\psi_h(z)$ that transforms as

$$\begin{aligned} \psi_h(z) \mapsto &= \left(\frac{dw}{dz}\right)^{(h+\delta_h)} \psi_h(w) \\ &= \left(\frac{dw}{dz}\right)^h \left[\psi_h(w) + \log\left(\frac{dw}{dz}\right) \phi_h(w)\right]. \end{aligned} \tag{5.3}$$

Here, δ_h is a nilpotent operator acting on local fields that satisfies $\delta_h^2 = 0$ and $\delta_h \psi_h = \phi_h$.

As an example, let us consider a rotation $z \mapsto w = e^{2\pi \mathrm{i}} z$.

For primary field,

$$e^{2\pi(L_0 - \bar{L}_0)} \phi_h(z) = \left(\frac{dw}{dz}\right)^h \phi_h(w) = e^{2\pi \mathrm{i} h} \phi_h(z), \tag{5.4}$$

where $L_0 - \bar{L}_0$ is the rotation generator. For logarithmic partner,

$$\begin{aligned} e^{2\pi(L_0 - \bar{L}_0)} \psi/h(z) &\mapsto e^{2\pi \mathrm{i}(h+\delta_h)} \psi_h(w) \\ &= e^{2\pi \mathrm{i} h} \left[\psi_h(w) + 2\pi \mathrm{i} \phi_h(w)\right]. \end{aligned} \tag{5.5}$$

$\rightarrow$ Action of rotation has a Jordan block structure!

(For audience familiar with CFT)

Logarithmic partners satisfy

$$T(z)\psi_h(w) \sim \frac{h\psi_h(w) + \phi_h(w)}{(z-w)^2} + \frac{\partial_w \psi_h(w)}{z-w}. \quad (5.6)$$

In terms of the Virasoro generators L_n, this means

$$L_0|\psi_h\rangle = h|\psi_h\rangle + |\phi_h\rangle, \quad L_n|\psi_h\rangle = 0, \quad \forall n \geq 1. \quad (5.7)$$

2pt correlations:

$$\langle \phi_h(z)\phi_h(w)\rangle = 0, \quad (5.8)$$

$$\langle \phi_h(z)\psi_h(w)\rangle = \frac{A}{(z-w)^{2h}}, \quad (5.9)$$

$$\langle \psi_h(z)\psi_h(w)\rangle = \frac{B - 2A\log(z-w)}{(z-w)^{2h}}. \quad (5.10)$$

Symplectic fermion theory Kausch (2000).:

$$S[\theta,\theta^*] = \frac{1}{4\pi}\int \mathrm{d}^d\boldsymbol{x}\,\partial_i\theta(\boldsymbol{x})\partial^i\theta^*(\boldsymbol{x}), \tag{5.11}$$

CFT in any dimension d.

Especially in 2D, symplectic fermion is a logarithmic CFT with the central charge $c=-2$.

Various models are described by this theory:

- Abelian sandpile model Piroux, Ruelle (2005)
- Haldane-Rezayi state Haldane, Rezayi (1988).
- Non-Hermitian Su-Schrieffer-Heeger model Chan, You, Wen, Ryu (2020)
- and many more...

3D symplectic fermion is studied in the context of dS/CFT correspondence Anninos et al. (2016).

This theory has logarithmic operators. $\theta(z,\bar{z})$ decomposes as $\theta(z,\bar{z}) = \theta(z) + \theta(\bar{z})$. Logarithmic partner of identity is given by

$$\omega(z) = {:}\theta(z)\theta^*(z){:} := \lim_{w\to z}[\theta(z)\theta^*(w) - \log(z-w)\mathbb{I}]. \tag{5.12}$$

This satisfies

$$e^{2\pi(L_0-\bar{L}_0)}\omega(z) = \omega(z) + 2\pi\mathrm{i}\mathbb{I}. \tag{5.13}$$

Local excitations:
$\langle\psi|\hat{H}_i|\psi\rangle = 0$ everywhere except for finite number of sites i.

Let us remove $\delta\hat{\psi}(\boldsymbol{x}_1)\delta\hat{\psi}^\dagger(\boldsymbol{x}_1)$ from $\hat{H}$:

$$\hat{H}' = t_+ \sum_{\tilde{\boldsymbol{x}}\in F} d\hat{\psi}^\dagger(\tilde{\boldsymbol{x}})d\hat{\psi}(\tilde{\boldsymbol{x}}) + t_- \sum_{\substack{\boldsymbol{x}\in V \\ \boldsymbol{x}\neq\boldsymbol{x}_1}} \delta\hat{\psi}(\boldsymbol{x})\delta\hat{\psi}^\dagger(\boldsymbol{x}). \tag{6.1}$$

If additional ground states appear, they correspond to local excitations at $\boldsymbol{x}_1$. Frustration-free conditions are given by

$$d\hat{\psi}(\boldsymbol{x})|\mathrm{GS}\rangle = 0, \quad \delta\hat{\psi}^\dagger(\boldsymbol{x})|\mathrm{GS}\rangle = 0 \ (\boldsymbol{x} \neq \boldsymbol{x}_1). \tag{6.2}$$

However, dropping one condition does not increase the ground states since $\{\delta\hat{\psi}^\dagger(\boldsymbol{x})\}$ are linearly dependent and satisfy

$$\delta\hat{\psi}^\dagger(\boldsymbol{x}_1) = -\sum_{\boldsymbol{x}\neq\boldsymbol{x}_1} \delta\hat{\psi}^\dagger(\boldsymbol{x}). \tag{6.3}$$

$\rightarrow$ No isolated excitations.

Next, remove the local terms $\delta\hat{\psi}(\boldsymbol{x})\delta\hat{\psi}^\dagger(\boldsymbol{x})$ at two points $\boldsymbol{x}_1, \boldsymbol{x}_2$. In this case, the additional ground state is given by

$$|\theta(\boldsymbol{x}_1)\theta(\boldsymbol{x}_2)\rangle := \frac{1}{\sqrt{Z}} \int \theta(\boldsymbol{x}_1)\theta(\boldsymbol{x}_2)|gd\theta\rangle \overleftarrow{\mathcal{D}}\theta. \tag{6.4}$$

Since the delta function is given as $\delta(\theta(\boldsymbol{x})) = \theta(\boldsymbol{x})$, this state represents that two punctures with the Dirichlet boundary condition are created at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Let us call these excitations Dirichlet excitations.

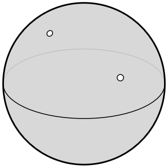


Figure 3: two punctures on a sphere

Anyonic excitations

The Dirichlet excitations are not created by local operators like $\hat{O}(\boldsymbol{x}_1)\hat{O}'(\boldsymbol{x}_2)$ from the ground state $|\xi\rangle$. Instead, they are created by non-local string operators as

$$
\begin{aligned}
|\theta(\boldsymbol{x}_1)\theta(\boldsymbol{x}_2)\rangle &= \frac{1}{\sqrt{Z}} \int (\theta(\boldsymbol{x}_1) - \theta(\boldsymbol{x}_2))\xi|gd\theta\rangle \overleftarrow{\mathcal{D}}\theta \\
&= \frac{1}{\sqrt{Z}} \int \int_{\boldsymbol{x}_2}^{\boldsymbol{x}_1} \mathrm{d}\theta\xi|gd\theta\rangle \overleftarrow{\mathcal{D}}\theta \\
&= \sqrt{4\pi} \int_{\boldsymbol{x}_2}^{\boldsymbol{x}_1} \hat{\psi}|\xi\rangle. \qquad (6.5)
\end{aligned}
$$

The curve connecting $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ can be continuously deformed since $d\hat{\psi}(\boldsymbol{x})|\xi\rangle = 0$.

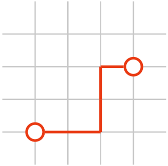


Figure 4: Two Dirichlet excitations created by a string operator

We can construct Neumann excitations by

$$|\phi^*(\tilde{\boldsymbol{x}}_1)\phi^*(\tilde{\boldsymbol{x}}_2)\rangle = \sqrt{4\pi}\int_{\tilde{\boldsymbol{x}}_2}^{\tilde{\boldsymbol{x}}_1} \star\widehat{\psi}^\dagger|\xi\rangle, \tag{6.6}$$

where ϕ^* is the dual field of θ^*.

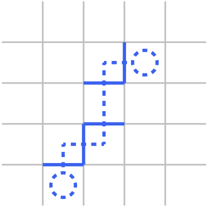


Figure 5: Two Neumann excitations created by a string operator

One way to see this duality is to exchange particles and holes in the definition of ground states.

$$|\tilde{\xi}\rangle = \frac{1}{\sqrt{Z}} \int \tilde{\xi} \exp(-g(\delta\phi^*, \hat{\psi}))|\tilde{0}\rangle \overleftarrow{\mathcal{D}}\phi^*, \tag{6.7}$$

where $|\tilde{0}\rangle$ is the state with all modes occupied. The quantum-classical correspondence for the dual field is given by

$$\hat{\psi}^\dagger \leftrightarrow \frac{\delta\phi^*}{\sqrt{4\pi}}, \quad \hat{\psi} \leftrightarrow \frac{\delta\phi}{\sqrt{4\pi}}. \tag{6.8}$$

The relation between θ, θ^* and ϕ, ϕ^* is expressed as

$$d\theta^\alpha = \delta\phi^\alpha, \quad (\theta^1, \theta^2) = (\theta, \theta^*), \; (\phi^1, \phi^2) = (\phi, \phi^*) \tag{6.9}$$

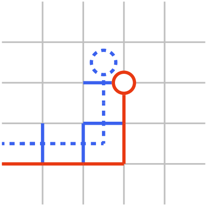
Using complex coordinates, the duality relation implies

$$\phi^\alpha(z, \bar{z}) = -i\theta^\alpha(z) + i\bar{\theta}^\alpha(\bar{z}) \quad \big(\theta^\alpha(z, \bar{z}) = \theta^\alpha(z) + \bar{\theta}^\alpha(\bar{z})\big)\,. \tag{6.10}$$

up to additive constant modes. c.f. T-duality

Anyonic excitations

Composite excitations:



In symplectic fermion theory, the composite excitation corresponds to the field

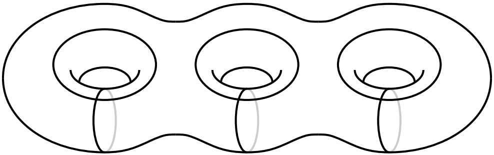
$$\begin{aligned} \phi^*\theta(z,\bar{z}) &= (-i\theta^*(z) + i\bar{\theta}^*(\bar{z}))(\theta(z) + \bar{\theta}(\bar{z})) \\ &= i\theta(z)\theta^*(z) - i\bar{\theta}(\bar{z})\bar{\theta}^*(\bar{z}) \\ &= i\omega(z) - i\bar{\omega}(\bar{z}), \end{aligned} \tag{6.11}$$

where ω and $\bar{\omega}$ are the logarithmic partners of the identity in the holomorphic and anti-holomorphic sectors, respectively.

If the spatial manifold has genus $g > 0$, there exist $2g$ non-contractible loops. String operators along such non-contractible loops are defined as

$$\hat{\Psi}_a := \oint_{\Gamma_a} \hat{\psi}, \quad \hat{\Psi}^\dagger_{\tilde{a}} := \oint_{\tilde{\Gamma}_a} \star\hat{\psi}^\dagger, \tag{6.12}$$

where $a = 1, \ldots, 2g$ labels independent non-contractible loops. Here, loops Γ_a and $\tilde{\Gamma}_b$ are chosen so that they intersect odd times if $a = b$ and even times if $a \neq b$.



The loop operators satisfy the relations

$$\{\hat{\Psi}_a, d\hat{\psi}(\tilde{\boldsymbol{x}})\} = \{\hat{\Psi}_a, \delta\hat{\psi}^\dagger(\boldsymbol{x})\} = 0, \tag{6.13}$$

$$\{\hat{\Psi}^\dagger_{\tilde{a}}, d\hat{\psi}(\tilde{\boldsymbol{x}})\} = \{\hat{\Psi}^\dagger_{\tilde{a}}, \delta\hat{\psi}^\dagger(\boldsymbol{x})\} = 0. \tag{6.14}$$

Therefore, loop operators preserve the frustration-free conditions and map ground states to ground states.

Note: These loop operators are not symmetries of the system since they do not commute with the Hamiltonian.

The anticommutation relations among the loop operators are calculated as

$$\{\hat{\Psi}_a, \hat{\Psi}^\dagger_{\tilde{b}}\} = \delta_{a,b}, \tag{6.15}$$

$$\{\hat{\Psi}_a, \hat{\Psi}_b\} = \{\hat{\Psi}^\dagger_{\tilde{a}}, \hat{\Psi}^\dagger_{\tilde{b}}\} = 0. \tag{6.16}$$

$\rightarrow 4^g$ -fold degeneracy on a genus g surface.

Spins

Let us consider the spin of the anyon-like excitations. The representation of 2π rotation is given by $e^{2\pi i(L_0-\bar{L}_0)}$. For the $\theta(z,\bar{z})$ field , we have

$$e^{2\pi i(L_0-\bar{L}_0)}\theta(z,\bar{z}) = e^{2\pi i(L_0-\bar{L}_0)}(\theta(z)+\bar{\theta}(\bar{z})) \\ = \theta(z,\bar{z}), \quad (6.17)$$

since $\theta(z)$ and $\bar{\theta}(\bar{z})$ are chiral primary fields with conformal weight $h=0$ and $\bar{h}=0$, respectively.

Similarly, a single ϕ^* excitation also has a trivial spin.

On the other hand, 2π rotation of the composite excitation $\phi^*\theta$ is given by

$$\begin{aligned} & e^{2\pi i(L_0-\bar{L}_0)}\phi^*\theta(z,\bar{z}) \\ & = ie^{2\pi i L_0}\omega(z) - ie^{-2\pi i\bar{L}_0}\bar{\omega}(\bar{z}) \\ & = i(1+2\pi i L_0+\cdots)\omega(z) - i(1-2\pi i\bar{L}_0+\cdots)\bar{\omega}(\bar{z}) \\ & = i(\omega(z)+2\pi i\mathbb{I}) - i(\bar{\omega}(\bar{z})-2\pi i\mathbb{I}) \\ & = \phi^*\theta(z,\bar{z}) - 4\pi\mathbb{I}. \end{aligned} \tag{6.18}$$

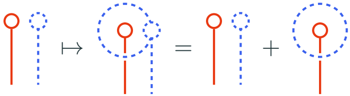
Therefore, when we rotate this anyon by 2π, it produces an additional term proportional to the identity operator, indicating a non-diagonalizable action of the rotation.

Spins

The same non-diagonalizable spin can be explicitly observed at the level of quantum states. Let us consider

$$4\pi \int_{\infty}^{\boldsymbol{x}} \star\hat{\psi}^{\dagger} \int_{\infty}^{\boldsymbol{x}} \hat{\psi}|\xi\rangle, \tag{6.19}$$

The action of 2π rotation is implemented by an anticlockwise winding $\phi^*(\boldsymbol{x})$ around $\theta(\boldsymbol{x})$.



(6.20)

This process yields additional contour integrals given as

$$4\pi \oint_{\boldsymbol{x}} \star\hat{\psi}^{\dagger} \int^{\boldsymbol{x}} \hat{\psi}|\xi\rangle = 4\pi \left\{ \oint_{\boldsymbol{x}} \star\hat{\psi}^{\dagger}, \int_{\infty}^{\boldsymbol{x}} \hat{\psi} \right\} |\xi\rangle = -4\pi|\xi\rangle. \tag{6.21}$$

$\rightarrow$ Consistent with the field theoretic calculation.

Summary and outlook

- I established the exact correspondence between QBT systems and symplectic fermion theory for both continuum and lattice models in any dimensions.
- In two dimension, I constructed anyonic excitations in QBT systems that come from the underlying symplectic fermion theory.
- Topological degeneracy is explained in terms of anyons.
- Observed non-diagonalizable action of rotation for the composite anyons.

Summary and outlook

Future directions:

- Study of the interaction-induced phases around the QBT systems from the viewpoint of symplectic fermion theory and anyonic excitations.
- Categorical formulation?
- Introduce spin structure and twist fields with $h = -1/8$.
- Entanglement properties
- PEPS representation of ground states