On Dynamic Critical Exponents of Gapless Frustration-free Systems

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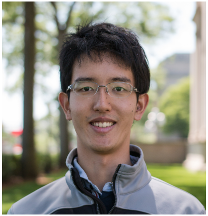
Collaborators



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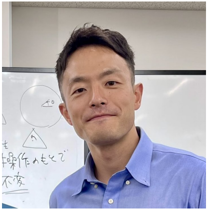

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(Univ. of Tokyo)

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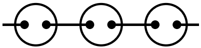
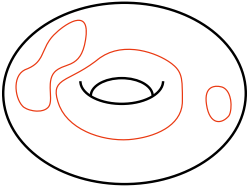
Introduction

Solvable models:

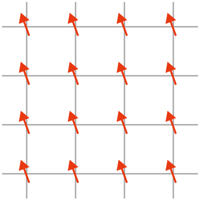
- Free fields, integrable models, conformal field theories
- Frustration-free (FF) systems



Affleck-Kennedy–
Lieb–Tasaki model



Toric code



ferromagnetic Heisenberg

Today's topic

Frustration-freeness serves as a characterization of gapless phases.

Definition 1. Frustration-freeness

A Hamiltonian H is called frustration-free (FF) if there exists a decomposition

$$H = \sum_i H_i + \text{const.} \tag{1.1}$$

such that the ground state (GS) minimizes each H_i simultaneously. We can assume $H_i \succeq 0$ (positive semidefinite). Then frustration-freeness is equivalent to

$$H_i|\text{GS}\rangle = 0, \quad \forall i. \tag{1.2}$$

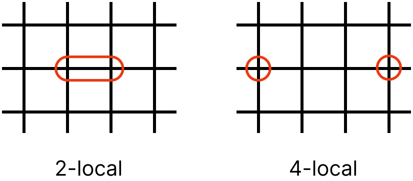
However, this definition is meaningless.

Trivial decomposition: $H = H$.

$\to$ Restrictions must be imposed on the decomposition of H.

Definition 2. k-Locality

We assume each H_i is k-local for a finite k, which means H_i acts non-trivially only on connected k sites.



Example of FF systems

■ 1+1D kinetic Ising model

Locally favored states:

$$|\psi_1\rangle := \frac{1}{\sqrt{\cosh(2\beta)}}(e^{\beta}|000\rangle + e^{-\beta}|010\rangle), \quad |\psi_2\rangle := \frac{1}{\sqrt{2}}(|001\rangle + |011\rangle), \tag{1.3}$$

$$|\psi_3\rangle := \frac{1}{\sqrt{\cosh(2\beta)}}(e^{\beta}|111\rangle + e^{-\beta}|101\rangle), \quad |\psi_4\rangle := \frac{1}{\sqrt{2}}(|110\rangle + |100\rangle). \tag{1.4}$$

- Local Hamiltonian: $H_i = \mathbb{1} - \sum_{n=1}^{4} |\psi_n\rangle\langle\psi_n|_{i-1,i,i+1}$ (3-local).
- Hamiltonian: $H = \sum_{i=1}^{L} H_i$
- GS (PBC): $|\mathrm{GS}\rangle \propto \sum_{\{\sigma\}} \exp\left(\frac{\beta}{2}\sum_i \sigma_i\sigma_{i+1}\right)|\{\sigma\}\rangle$.
- Schmidt decomposition: $|\mathrm{GS}\rangle = \sum_{n=1}^{4} \lambda_n|\psi_n\rangle_{i-1,i,i+1} \otimes |\phi_n\rangle_{\Lambda\setminus\{i-1,i,i+1\}}$

Remark

Determining whether a given state is a GS becomes easier in FF cases (if we already have a nice decomposition).

Examples of FF systems have explicit form of the GS for this reason.

In general, it is computationally hard to determine whether a given Hamiltonian is FF.

- If the decomposition is specified, it is a QMA_1-hard problem.
  Bravyi, arXiv:quant-ph/0602108
- There is a polynomial-time algorithm to search a nice decomposition (with looser restrictions on decomposition than k-locality.)
  Takahashi, Rayudu, Zhou, King, Thompson, Parekh, arXiv:2307.15688

Remark

Non-trivial FF systems need degeneracy of locally favored states.

Let us consider

$$H = H_{12} \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes H_{23}, \tag{1.5}$$

where

$$H_{12} = \mathbb{1} - |\psi_{12}\rangle\langle\psi_{12}|, \quad H_{23} = \mathbb{1} - |\psi_{23}\rangle\langle\psi_{23}|. \tag{1.6}$$

If H is FF under this decomposition,

$$|\mathrm{GS}\rangle = |\psi_{12}\rangle \otimes |\phi_3\rangle = |\phi_1\rangle \otimes |\psi_{23}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle. \tag{1.7}$$

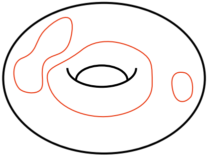
Thus GS must be a trivial tensor product state.

FF-ness is unstable under general perturbations.

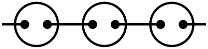
Gapped FF systems vs Gapless FF systems

FF Hamiltonians can approximate general gapped quantum phases.

- Many representative models of gapped phases.



Toric code: $\mathbb{Z}_2$ topological order



AKLT model: Haldane phase

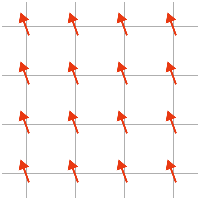
- The GS of a gapped Hamiltonian is also the GS of a (superpolynomially local) FF Hamiltonian. Kitaev, Ann. Phys. 321(1), 2-111 (2006).

Gapped FF systems vs Gapless FF systems

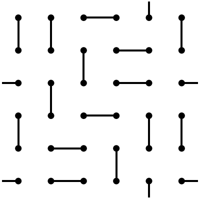
However, gapless FF systems exhibit different low-energy behaviors than typical gapless systems (as we will see).

FF gapless systems are useless as an approximation of gapless systems.

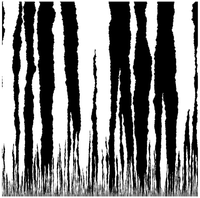
$\leftrightarrow$ FF gapless systems are interesting in their own right.



ferromagnetic Heisenberg



Rokhsar–Kivelson point



critical kinetic Ising

Dynamic critical exponents

We focus on dynamic critical exponents.

Definition 3. Spectral gap

Let us take the ground state energy of H to be zero. The spectral gap $\mathrm{gap}(H)$ is the smallest nonzero eigenvalue of H.

Definition 4. Dynamic critical exponent

For gapless systems, the dynamic critical exponent z is defined by

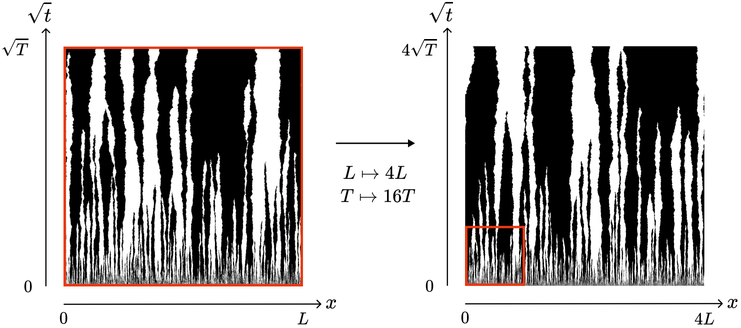
$$\mathrm{gap}(H) \sim L^{-z} \tag{2.1}$$

where L is the linear size of the system.

- Typical gapless systems : $z = 1$
- FF gapless systems : $z \geq 2$ (No complete proof)

Critical points with z are expected to have invariance under the Lifshitz scale transformation given by

$$\boldsymbol{x} \mapsto \lambda \boldsymbol{x}, \quad t \mapsto \lambda^z t, \quad (\lambda > 0). \tag{2.2}$$



Lifshitz scale invariance of the zero-temp. kinetic Ising model ($z = 2$).

Dynamic critical exponents

Gapless systems with z are expected to have the dispersion relation

$$E_k \sim k^z. \tag{2.3}$$

Conjecture: gapless FF systems have quadratic or softer dispersion.

Masaoka, Soejima, Watanabe, PRB 110, 195140 (2024)

- Coleman's theorem in the contexts of relativistic field theory: Spontaneous symmetry breaking (SSB) of continuous symmetries does not occur in 1+1D systems at $T = 0$.

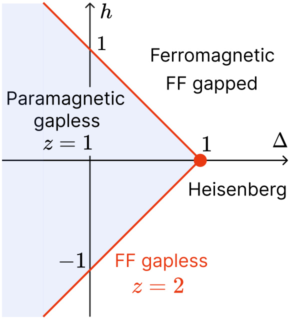
  Coleman, Commun.Math. Phys. 31, 259–264 (1973).
- However, it can occur in 1+1D gapless FF systems because of the quadratic of softer dispersions. Watanabe, Katsura, Lee, PRL 133, 176001 (2024)

Case study: XXZ model + magnetic field

gapless FF $\Rightarrow z \geq 2$

Let us check $z \geq 2$ for gapless FF systems in specific examples.

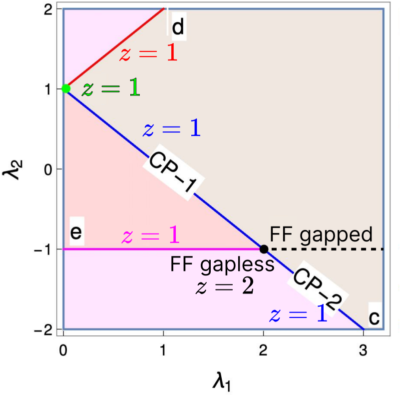
$$H = -\sum_{i=1}^{L}(X_iX_{i+1} + Y_iY_{i+1} + \Delta Z_iZ_{i+1}) + 2h\sum_{i=1}^{L} Z_i + \text{const.}, \qquad (2.4)$$



XXZ model with a magnetic field. For example, see the textbook by Franchini (2017).

Case study: quantum Ising model + cluster interaction

$$H = -\sum_{i=1}^{L} (\lambda_1 Z_i Z_{i+1} + \lambda_2 Z_{i-1} X_i Z_{i+1}) + \sum_{i=1}^{L} X_i + \text{const.} \tag{2.5}$$



from Kumar, Kartik, Rahul, Sarkar, Sci. Rep. 11, 1004 (2021). modified

There are proofs of $z \geq 2$ in the case of open boundary condition.

Gosset, Mozgunov, J. Math. Phys. 57, 091901 (2016). Anshu, PRB 101, 165104 (2020).

Lemm, Xiang, J. Phys. A: Math. Theor. 55 295203 (2022).

These results do not give a rigorous bound for the bulk modes since there can be edge modes in OBC.

arXiv > cond-mat > arXiv:2406.06415

Condensed Matter > Strongly Correlated Electrons

[Submitted on 10 Jun 2024]

Rigorous lower bound of dynamic critical exponents in critical frustration-free systems

Rintaro Masaoka, Tomohiro Soejima, Haruki Watanabe

We show that $z \geq 2$ for a wide range of FF gapless models without assuming any boundary conditions (but assuming additional assumptions).

Gosset–Huang inequality

The techniques needed for the proof had already established.

Theorem 1. Gosset–Huang inequality Gosset, Huang, PRL 116, 097202. (2016)

Let H be an FF Hamiltonian and

- G : Projector onto the ground space,
- $\mathcal{O}_{\boldsymbol{x}}, \mathcal{O}'_{\boldsymbol{y}}$: Local operators

Then

$$\frac{|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1}-G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle|}{\|\mathcal{O}^{\dagger}_{\boldsymbol{x}}|\mathrm{GS}\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle\|} \leq 2\exp\left(-C|\boldsymbol{x}-\boldsymbol{y}|\sqrt{\mathrm{gap}(H)}\right), \tag{2.6}$$

where C is a positive constant.

(Gosset and Huang were aware of the application to the gapless FF systems, but they did not demonstrate the scope of its applicability.)

$z \geq 2$ from Gosset–Huang inequality

Definition 5. "Critical" FF systems

We say that an FF system is critical, if there exists a correlation function such that

$$|\boldsymbol{x} - \boldsymbol{y}| \sim L \quad \text{and} \quad \frac{|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1} - G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle|}{\|\mathcal{O}_{\boldsymbol{x}}^\dagger|\mathrm{GS}\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle\|} \gtrsim \frac{1}{L^\Delta}, \ (\Delta > 0). \quad (2.7)$$

Corollary 1. <u>Masaoka</u>, Soejima, Watanabe arXiv:2406.06415.

Critical FF systems satisfy $z \geq 2$.

Proof: From the Gosset–Huang inequality,

$$\frac{1}{L^\Delta} \lesssim \frac{|\langle \mathrm{GS}|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1} - G)\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle|}{\|\mathcal{O}_{\boldsymbol{x}}^\dagger|\mathrm{GS}\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\mathrm{GS}\rangle\|} \leq 2\exp\left(-CL\sqrt{\mathrm{gap}(H)}\right). \quad (2.8)$$

This inequality breaks for sufficiently large L if $z < 2$. □

$z \geq 2$ from Gosset–Huang inequality

Critical FF systems satisfy $z \geq 2$.

Our argument is highly general because we do not assume

- boundary condition
- spatial dimension
- structure of the lattice
- translational invariance

Also, our result can be extended to fermionic FF systems.

(Of course, we should explicitly construct an algebraic correlation function.)

Our result: $z \geq 2$ for dynamic critical phenomena

We also prove $z \geq 2$ for dynamic critical phenomena, leaving the contexts of quantum systems.

Critical points	z (numerical)	References
Ising (2D)	$2.1667(5) \geq 2$	Nightingale, Blöte, PRB 62, 1089 (2000).
Ising (3D)	$2.0245(15) \geq 2$	Hasenbusch, PRE 101, 022126 (2020).
Heisenberg (3D)	$2.033(5) \geq 2$	Astillero, Ruiz-Lorenzo, PRE 100, 062117 (2019).
three-state Potts (2D)	$2.193(5) \geq 2$	Murase, Ito, JPSJ 77, 014002 (2008).
four-state Potts (2D)	$2.296(5) \geq 2$	Phys. A: Stat. Mech. Appl. 388, 4379 (2009).

Dynamic critical exponents of Markov processes relaxing to critical equilibrium states.

We focus on a specific class of FF Hamiltonians.

Definition 6. (Generalized) Rokhsar–Kivelson Hamiltonian

$H^{\mathrm{RK}} = \sum_i H_i^{\mathrm{RK}}$ is a (generalized) RK Hamiltonian if

1. Hamiltonian is FF
2. GS is written as

$$|\Psi_{\mathrm{RK}}\rangle = \sum_{\mathcal{C}} \sqrt{\frac{w(\mathcal{C})}{\mathcal{Z}}}\,|\mathcal{C}\rangle, \quad \mathcal{Z} = \sum_{\mathcal{C}} w(\mathcal{C}), \tag{3.1}$$

   where $w(\mathcal{C})$ is a Boltzmann weight of a classical statistical system.
3. The off-diagonal elements of H_i are non-positive

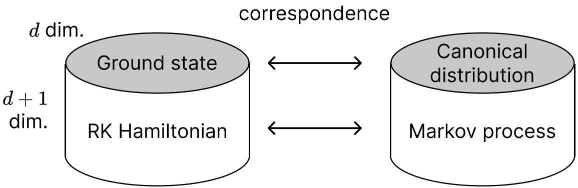
There are several names for this class: sto**qu**astic FF Hamiltonian, stochastic matrix form, stochastic quantization.

Correspondence between RK Hamiltonians and Markov processes

RK Hamiltonians correspond to Markov processes with local state updates and the detailed balance condition.

Henley, J. Phys.: Condens. Matter 16 S891 (2004).

Castelnovo *et al.*, Ann. Phys. 318, 316 (2005).



Correspondence between RK Hamiltonians and Markov processes.

The correspondence is explicitly given by

$$(W_i)_{\mathcal{C}\mathcal{C}'} := -\sqrt{w(\mathcal{C})}\,(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'}\,\frac{1}{\sqrt{w(\mathcal{C}')}}. \tag{3.2}$$

$W := \sum_i W_i$ is the transition-rate for the corresponding Markov process.

Correspondense between RK Hamiltonians and Markov processes

Imaginary-time Schrödinger eq. $\mathrm{d}\lvert\psi(t)\rangle/\mathrm{d}t = -H^{\mathrm{RK}}\lvert\psi(t)\rangle$	Master eq. $\mathrm{d}p(t)/\mathrm{d}t = W p(t)$
Ground state $\lvert\Psi_{\mathrm{RK}}\rangle = \sum_{\mathcal{C}} \sqrt{w(\mathcal{C})/\mathcal{Z}}\,\lvert\mathcal{C}\rangle$	Steady state $p_{\mathrm{eq}}(\mathcal{C}) = w(\mathcal{C})/\mathcal{Z}$
Symmetricity $(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'} = (H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'}$	Detailed balance condition $(W_i)_{\mathcal{C}\mathcal{C}'}w(\mathcal{C}') = (W_i)_{\mathcal{C}'\mathcal{C}}w(\mathcal{C})$
FF-ness $\langle\Psi_{\mathrm{RK}}\rvert H_i^{\mathrm{RK}} = 0$	Probability conservation $\sum_{\mathcal{C}}(W_i)_{\mathcal{C}\mathcal{C}'} = 0$
Dynamic critical exponent $\mathrm{gap}(H^{\mathrm{RK}}) \sim L^{-z}$	Dynamic critical exponent $\tau := 1/\,\mathrm{gap}(-W) \sim L^{z}$

■ 2+1D kinetic Ising model (Gibbs sampling)

Boltzmann weight:

$$w(\mathcal{C}) = \exp\left(\beta \sum_{\langle i,j\rangle} \sigma_i \sigma_j\right) \quad (\sigma_i = \pm 1). \tag{3.3}$$

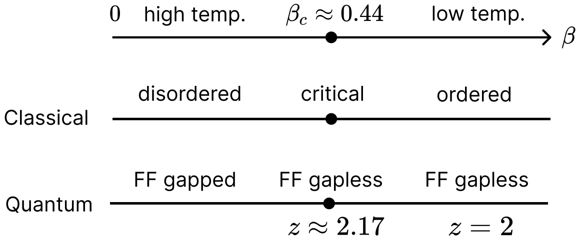
The Gibbs sampling (heat bath) algorithm is given by

$$(W_i)_{\mathcal{C}'\mathcal{C}} = -(W_i)_{\mathcal{C}\mathcal{C}} = \frac{w(\mathcal{C}')}{w(\mathcal{C}) + w(\mathcal{C}')}, \tag{3.4}$$

where $|\mathcal{C}'\rangle := \sigma_i^x |\mathcal{C}\rangle$. We do not assume any conserved quantity (model A). The corresponding RK Hamiltonian is

$$H_i^{\mathrm{RK}} = \frac{1}{2\cosh(\beta \sum_{j\sim i} Z_j)} \left(\mathrm{e}^{-\beta Z_i \sum_{j\sim i} Z_j} - X_i\right). \tag{3.5}$$

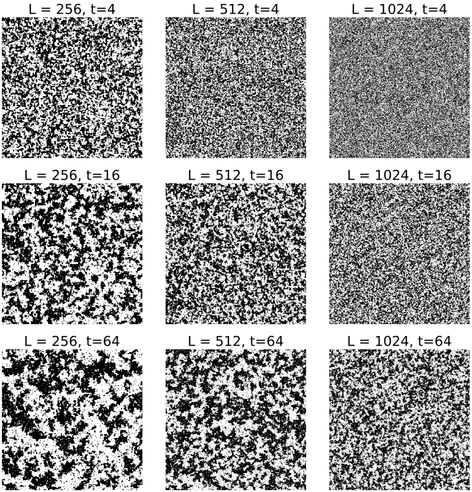
The quantum phase diagram is obtained from the classical phase diagram.



We focus on the critical point (ordered phase is another interesting topic).

Example: 2+1D kinetic Ising model

At $\beta = \beta_c \approx 0.44$, the relaxation time diverges as $L \to \infty$. ($z \approx 2.17$)



Markov Chain Monte Carlo simulation for 2+1D kinetic Ising model

Dynamic critical exponents for various critical points

Critical points	z (numerical)	References
Ising (2D)	$2.1667(5) \geq 2$	Nightingale, Blöte, PRB 62, 1089 (2000).
Ising (3D)	$2.0245(15) \geq 2$	Hasenbusch, PRE 101, 022126 (2020).
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Dynamic critical exponents of RK Hamiltonians of critical points

RK Hamiltonians of critical points, called conformal quantum critical points (CQCP), seemed to satisfy $z \geq 2$.

- Conjectured in Isakov, Fendley, Ludwig, Trebst, Troyer, PRB 83, 125114 (2011).
- Previous rigorous result: $z \geq 2 - \eta$. Halperin, PRB 8, 4437 (1973).

$z \geq 2$ for conformal quantum critical points

Theorem 2. Masaoka, Soejima, Watanabe arXiv:2406.06415.

RK Hamiltonians of critical points (CQCPs) satisfy $z \geq 2$.

Our framework: If there is a correlation function such that

$$|\boldsymbol{x} - \boldsymbol{y}| \sim L, \quad \frac{|\langle \Psi | \mathcal{O}_{\boldsymbol{x}} (\mathbb{1} - G) \mathcal{O}'_{\boldsymbol{y}} | \Psi \rangle|}{\| \mathcal{O}^{\dagger}_{\boldsymbol{x}} | \Psi \rangle \| \| \mathcal{O}'_{\boldsymbol{y}} | \Psi \rangle \|} \gtrsim \frac{1}{L^{\Delta}}, \tag{3.6}$$

then $z \geq 2$.

$z \geq 2$ for conformal quantum critical points

Let us explicitly construct an algebraic correlation function to prove $z \geq 2$.

Quantum classical correspondence for a diagonal operator $O(\mathcal{C})\delta_{\mathcal{C}\mathcal{C}'}$:

$$\langle \Psi_{\mathrm{RK}} | O | \Psi_{\mathrm{RK}} \rangle = \sum_{\mathcal{C}} \frac{O(\mathcal{C}) w(\mathcal{C})}{\mathcal{Z}} =: \langle O \rangle. \tag{3.7}$$

There is an operator O_i such that

$$\langle O_i \rangle = 0, \quad \langle O_i^2 \rangle = \mathrm{const.}, \quad \langle O_i O_j \rangle \sim \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^{2\Delta_O}}, \tag{3.8}$$

where Δ_O is the scaling dimension of O_i. Thus, if $|\boldsymbol{x}_i - \boldsymbol{x}_j| \sim L$,

$$\frac{|\langle \Psi_{\mathrm{RK}} | \mathcal{O}_i (\mathbb{1} - G) \mathcal{O}_j | \Psi_{\mathrm{RK}} \rangle|}{\| \mathcal{O}_i | \Psi_{\mathrm{RK}} \rangle \| \| \mathcal{O}_j | \Psi_{\mathrm{RK}} \rangle \|} = \frac{|\langle \mathcal{O}_i \mathcal{O}_j \rangle - \langle \mathcal{O}_i \rangle \langle \mathcal{O}_j \rangle|}{\langle \mathcal{O}_i^2 \rangle} \sim L^{-2\Delta_{\mathcal{O}}}. \tag{3.9}$$

Here, we assumed $G = |\Psi_{\mathrm{RK}}\rangle\langle\Psi_{\mathrm{RK}}|$ for simplicity.

Therefore, $z \geq 2$.

No-go theorem for local MCMC methods with detailed balance

Rephrasing the theorem in the language of Markov processes, we obtain the following no-go theorem.

No-go theorem

Markov processes for critical points with local state updates and the detailed balance condition satisfy $z \geq 2$.

$\rightarrow$ First proof of an empirical fact known in the contexts of dynamic critical phenomena.

Remark

We can consider more general ground states with a phase factor:

$$|\mathrm{GS}\rangle = \sum_{\mathcal{C}} \mathrm{e}^{\mathrm{i}\theta(\mathcal{C})} \sqrt{\frac{w(\mathcal{C})}{\mathcal{Z}}}\, |\mathcal{C}\rangle, \quad \theta(\mathcal{C}) \in \mathbb{R}. \tag{3.10}$$

■ Fine-tuned Fibonacci Levin Wen model

Fendley, Fradkin, PRB 72, 024412 (2005)., Fendley, Ann. Phys. 323(12), 3113-3136 (2008).

- $w(\mathcal{C})$ represents $c = 14/15$ CFT.
- GS shows algebraic correlations.
- It cannot be mapped to a Markov process due to the sign problem.

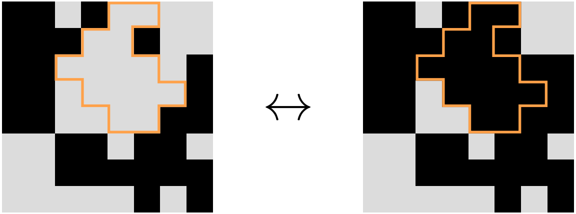
We can show $z \geq 2$ also in this case since phases $\pm\theta(\mathcal{C})$ cancel in correlation functions of diagonal operators.

Stochastic dynamics with $z < 2$

By violating the assumptions in the no-go theorem, one can create Markov processes with faster relaxation with $z < 2$.

■ Wolff cluster algorithm Wolff, PRL. 62, 361 (1988).

Locality: $\times$, Detailed balance condition: ✓



State update of the Wolff cluster algorithm

$z \approx 0.3$ for the 2D Ising critical point. Liu *et al.* PRB 89, 054307 (2014).

■ Asymmetric simple exclusion process (ASEP)

Locality: ✓, Detailed balance condition: ×

XXZ model with a non-Hermitian term:

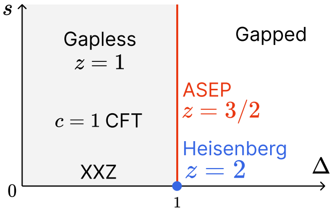
$$H_i = \frac{1}{4}(1 - \Delta Z_i Z_{i+1}) - \frac{1+s}{2}\sigma_i^+ \sigma_{i+1}^- - \frac{1-s}{2}\sigma_i^- \sigma_{i+1}^+ + \frac{s}{2}(Z_i - Z_{i+1}) \quad (3.11)$$

$\Delta < 1$: Gapless phase ($z = 1$)
$\Delta > 1$: Gapped phase
$\Delta = 1$: Stochastic line

- $s = 0$: Heisenberg ($z = 2$, EW class)
- $s > 0$: ASEP ($z = 3/2$, KPZ class)

  Kim, PRE 52, 3512 (1995).

  Gwa, Spohn, PRA 46, 844 (1992).



Phase diagram of XXZ model with a non-Hermitian term.

FF models are expected to flow into FF effective field theories.

Definition 7. Frustration-free field teory (FFFT)

A field theory is FF if the Hamiltonian density $\mathcal{H}(x)$ is positive semi-definite and

$$\forall x,\ \mathcal{H}(x)|\mathrm{GS}\rangle = 0. \tag{4.1}$$

In the following slides, we look at some examples of FF field theories.

Topological quantum field theories are FF.

e.g. Chern–Simons theory:

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int \mathrm{d}t\, \mathrm{d}^2 x\, \varepsilon^{\mu\nu\lambda}\, \mathrm{Tr}\left[A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right]. \tag{4.2}$$

Hamiltonian density:

$$\mathcal{H}(x) = e^2\, \mathrm{Tr}\left[E^\dagger(x) E(x) \right], \quad E(x) = \frac{\delta}{\delta A_z(x)} - \frac{k}{4\pi} A_{\bar{z}}(x). \tag{4.3}$$

GS wave functional $\Psi[A]$ satisfies $E(x)\Psi[A] = 0 \Rightarrow$ FF.

Another derivation

$$\mathcal{H}(x) = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{CS}}[A, g]}{\delta g_{00}(x)} = 0. \tag{4.4}$$

Leeh-Schlieder theorem

Relativistic field theories satisfy

$$\mathcal{O}(x)|\mathrm{GS}\rangle = 0 \Rightarrow \mathcal{O}(x) = 0, \tag{4.5}$$

where $\mathcal{O}(x)$ is a local operator.

Corollary

Relativistic field theories are not FF except for the case of $\mathcal{H}(x) = 0$.

We can construct the $d+1$-dim. FFFT from a d-dim. field theory by stochastic quantization ($\approx$ RK Hamiltonians).

Parisi, Wu, Sci. sin, 24(4), 483-496, (1981), Dijkgraaf, Orlando, Reffert, arxiv:0903.0732 (2009)

Let us consider the following master equation (Fokker–Planck equation).

$$\begin{aligned}\frac{\partial}{\partial t} P[\phi, t] &= W P[\phi, t] \\ &= \frac{1}{2} \int \mathrm{d}^d x \, \frac{\delta}{\delta \phi(x)} \left(\frac{\delta S_{\mathrm{cl}}}{\delta \phi(x)} + \frac{\delta}{\delta \phi(x)} \right) P[\phi, t], \end{aligned} \tag{4.6}$$

where

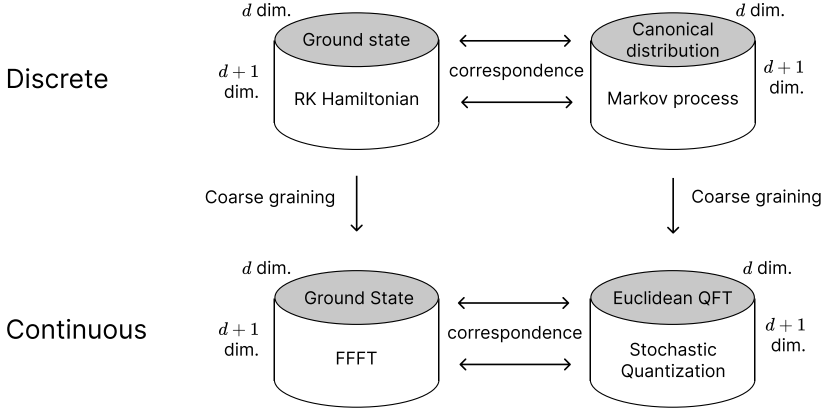
- $P[\phi, t]$ is a probability distribution,
- $S_{\mathrm{cl}}[\phi]$ is the action of an Euclidean field theory.

Correspondence between Hamiltonian and transition-rate:

$$H = -\frac{1}{\sqrt{\mathrm{e}^{-S_{\mathrm{cl}}}}} W \sqrt{\mathrm{e}^{-S_{\mathrm{cl}}}} = \int \mathrm{d}^d x \, \mathcal{H}(x), \tag{4.7}$$

where

$$\mathcal{H}(x) = \frac{1}{2} \mathcal{Q}^\dagger(x) \mathcal{Q}(x), \quad \mathcal{Q}(x) := \frac{\delta}{\delta\phi(x)} + \frac{1}{2} \frac{\delta S_{\mathrm{cl}}}{\delta\phi(x)}. \tag{4.8}$$


Discrete
Continuous
d dim.
$d + 1$ dim.
Ground state
RK Hamiltonian
correspondence
Canonical distribution
Markov process
Coarse graining
Coarse graining
Ground State
FFFT
correspondence
Euclidean QFT
Stochastic Quantization

We can construct the $d+1$-dim. gapless FFFT from a d-dim. CFT.
These theories are considered to be the effective field theories of CQCPs (RK Hamiltonians of critical points).

Our results provide microscopic proof of $z \geq 2$ for the stochastic quantization of a CFT.
However, macroscopic understanding is still lacking.

Summary

Our study highlights the unique nature of the gapless FF systems. We have established $z \geq 2$ for dynamic critical exponents of various FF systems:

- Conformal quantum critical points. (Stochastic quantization of CFT)
- FF systems with a plane-wave ground state.
- FF systems with a hidden correlation.

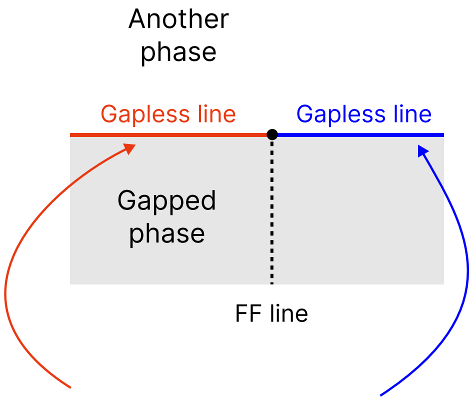
Also, we established $z \geq 2$ for Markov processes with locality and detailed balance condition.

Open questions

Complete proof of $z \geq 2$ for gapless FF systems.

Is there a macroscopic proof of $z \geq 2$?

How fast does non-Hermiticity (breaking detailed balance) speed up relaxation?



Transition between what and what?

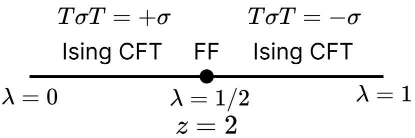
An interesting example is in Verresen *et al.*, PRX 11, 041059 (2021).

$$H = -\sum_i (Z_i Z_{i+1} + X_i) \tag{5.1}$$

$$H' = -\sum_i (Y_i Y_{i+1} + X_i) \tag{5.2}$$

$$H(\lambda) = \lambda H + (1-\lambda)H' \quad (0 \le \lambda \le 1). \tag{5.3}$$

This interpolation preserves $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ symmetry.



THANK YOU.