Rigorous lower bound of dynamic critical exponents in critical frustration-free systems

(arXiv:2406.06415)

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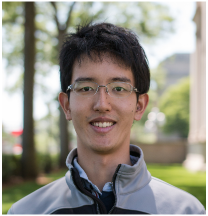

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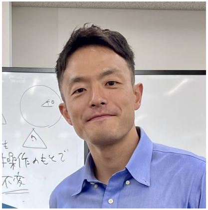

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Introduction

Quantum many-body systems are notoriously hard to solve. One way to gain qualitative insights is to start with exactly solvable models and uncover the underlying universal physics.

- Free field theories
- Conformal field theories (CFT)
- Bethe ansatz

The topic of today's talk, frustration-free systems, can also be seen as part of this class of solvable models. However, their solvability is relatively limited: while the ground state can be explicitly written down, determining the excited states is generally difficult.

Definition 1. Frustration-freeness

A Hamiltonian H is called frustration-free (FF) if and only if there exists a decomposition

$$H = \sum_i H_i + E_0 \mathbb{1}, \quad E_0 \in \mathbb{R}, \tag{1.1}$$

and the following conditions hold.

- Each local Hamiltonian H_i is positive semidefinite with a zero eigenvalue.
- There is a ground state (GS) $|\Psi\rangle$ such that $H_i|\Psi\rangle = 0$ for all H_i.

Definition 2. Locality

In this talk, we assume each H_i is k-local for a finite k, which means H_i acts nontrivially only on connected k sites.

X_i, Y_i, Z_i: Pauli matrices at site i.

■ $d+1$D spin-$1/2$ ferromagnetic Heisenberg model

Let Λ be a d-dimensional lattice. We consider $s=1/2$ spins on vertices of Λ. The Hamiltonian is given by

$$H = \sum_{\langle i,j\rangle} H_{i,j}, \quad \text{where } \langle i,j\rangle \text{ is a pair of adjacent vertices,} \tag{1.2}$$

$$H_{i,j} = \frac{1}{4}(\mathbb{1} - X_iX_j - Y_iY_j - Z_iZ_j) \geq 0, \tag{1.3}$$

$$\ker H_i = \mathrm{Span}\{|00\rangle, |11\rangle, |01\rangle + |10\rangle\}. \tag{1.4}$$

Ground states:

$$|\Psi_N\rangle = \frac{1}{\sqrt{\mathcal{Z}_N}} \sum_{\{n_i\}} \delta\left(\sum_{i\in\Lambda} n_i - N\right) |\{n_i\}_{i\in\Lambda}\rangle, \tag{1.5}$$

where $\mathcal{Z}_N$ is the normalization constant. These ground states satisfy $H_i|\Psi_N\rangle = 0$, thus this model is FF.

Examples of FF systems

■ Toric code Kitaev Ann. Phys. 303, 2 (2003).

Consider $s = 1/2$ spins at the edges of a square lattice.

The Hamiltonian is given by

$$H = \sum_{v \in \mathrm{Vertices}} (\mathbb{1} - A_v) + \sum_{f \in \mathrm{Faces}} (\mathbb{1} - B_f), \tag{1.6}$$

$$A_v = \prod_{l \in \delta v} Z_l, \quad B_f = \prod_{l \in \partial f} X_l \tag{1.7}$$

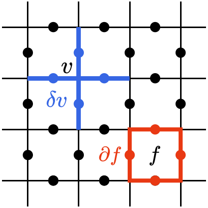

Figure 1: Interactions of the toric code.

All terms $\{A_v, B_f\}$ commute with each other.

Simultaneous diagonalization of $\{A_v, B_f\}$ yields a GS such that

$$(\mathbb{1} - A_v)|\Psi\rangle = (\mathbb{1} - B_f)|\Psi\rangle = 0. \tag{1.8}$$

Thus, this model is FF.

Note that commuting local Hamiltonians does not imply FF-ness.

Can FF Hamiltonians describe universal properties of quantum phases?

- Yes, for many gapped phases.
  - Examples: Toric code, Affleck-Kennedy–Lieb–Tasaki model, etc.
  - GS of gapped Hamiltonian is GS of some (superpolynomially) local FF Hamiltonian. Kitaev, Ann. Phys. 321(1), 2-111 (2006).
- No, for typical gapless phases (with emergent Lorentz symmetry).
  - FF gapless systems often exhibit different low-energy behaviors than typical gapless systems (as we will see).

Dynamic ciritcal exponents of FF gapless systems

We focus on dynamic critical exponents.

Definition 3. Spectral gap

Let H be a positive semidefinite matrix with a zero eigenvalue. The spectral gap $\mathrm{gap}(H)$ is the smallest nonzero eigenvalue of H.

Definition 4. Dynamic critical exponent

For gapless systems, the dynamic critical exponent z is defined by

$$\mathrm{gap}(H) \sim L^{-z} \tag{1.9}$$

where L is the linear dimension of the system.

- Typical gapless systems : $z = 1$
- FF gapless systems : $z \geq 2$ (no complete proof)

Case study

The dynamic critical exponent z is defined by $\mathrm{gap}(H) \sim L^{-z}$.

$$H = -\sum_{i=1}^{L}(X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}) + 2h \sum_{i=1}^{L} Z_i + \text{const.}, \tag{2.1}$$

$$H = -\sum_{i=1}^{L}(\lambda_1 Z_i Z_{i+1} + \lambda_2 Z_{i-1} X_i Z_{i+1}) + \sum_{i=1}^{L} X_i + \text{const.} \tag{2.2}$$

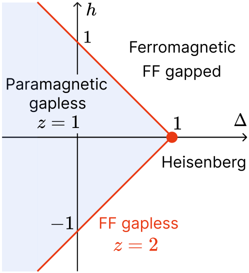

Figure 2: XXZ model with a magnetic field. For example, see the textbook by Franchini (2017).

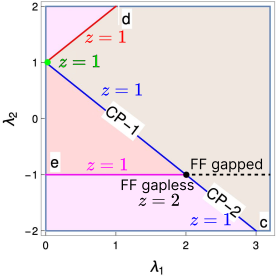

Figure 3: Kumar *et al.*, Sci Rep 11, 1004 (2021)., modified

Our result

There are proofs of $z \geq 2$ in the case of open boundary condition.

Gosset, Mozgunov, J. Math. Phys. 57, 091901 (2016). Anshu, PRB 101, 165104 (2020).

Lemm, Xiang, J. Phys. A: Math. Theor. 55 295203 (2022).

These results do not give a rigorous bound for the bulk modes since there can be edge modes in OBC.

We show that $z \geq 2$ for a wide range of FF gapless models without assuming any boundary conditions (but assuming additional assumptions).

arXiv > cond-mat > arXiv:2406.06415

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Condensed Matter > Strongly Correlated Electrons

[Submitted on 10 Jun 2024]

Rigorous lower bound of dynamic critical exponents in critical frustration-free systems

Rintaro Masaoka, Tomohiro Soejima, Haruki Watanabe

The dynamic critical exponent z characterizes the finite-size gap in gapless quantum many-body systems. We establish a rigorous lower bound $z \geq 2$ for frustration-free Hamiltonians on any lattice in any spatial dimension, given that their ground state exhibits a power-law decaying correlation function. This bound applies to representative classes of frustration-free Hamiltonians, including Rokhsar-Kivelson Hamiltonians, which are in one-to-one correspondence to Markov chains with locality, as well as parent Hamiltonians of critical projected entangled pair states with either a unique ground state or topologically degenerate ground states, and Hamiltonians with a plane-wave ground state.

Comments: 16 pages, 4 figures, 2 tables

Subjects: **Strongly Correlated Electrons (cond-mat.str-el)**; Other Condensed Matter (cond-mat.other); Statistical Mechanics (cond-mat.stat-mech); Quantum Physics (quant-ph)

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(or arXiv:2406.06415v1 **[cond-mat.str-el]** for this version)

https://doi.org/10.48550/arXiv.2406.06415

Our proof relies on the following inequality.

Theorem 1. Gosset–Huang inequality Gosset, Huang, PRL 116, 097202. (2016)

Let H be an FF Hamiltonian and

- $|\Psi\rangle$: Ground state of H,
- G : Projector onto the ground subspace,
- ϵ : Spectral gap of H,
- $\mathcal{O}_{\boldsymbol{x}}, \mathcal{O}'_{\boldsymbol{y}}$: Local operators on the positions $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

Then

$$\frac{|\langle\Psi|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1}-G)\mathcal{O}'_{\boldsymbol{y}}|\Psi\rangle|}{\|\mathcal{O}^{\dagger}_{\boldsymbol{x}}|\Psi\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\Psi\rangle\|} \leq 2\exp\left(-\text{const.}\times|\boldsymbol{x}-\boldsymbol{y}|\sqrt{\epsilon}\right). \tag{2.3}$$

Definition 5. “Criticality” for FF systems

We say that an FF Hamiltonian is critical, if there exists a correlation function such that

$$|\boldsymbol{x} - \boldsymbol{y}| \sim L \quad \text{and} \quad \frac{|\langle \Psi | \mathcal{O}_{\boldsymbol{x}} (\mathbb{1} - G) \mathcal{O}'_{\boldsymbol{y}} | \Psi \rangle|}{\| \mathcal{O}^{\dagger}_{\boldsymbol{x}} | \Psi \rangle \| \| \mathcal{O}'_{\boldsymbol{y}} | \Psi \rangle \|} \gtrsim L^{-\Delta}, \tag{2.4}$$

where Δ is a positive number.

Corollary 1. **Masaoka, Soejima, Watanabe** arXiv:2406.06415.

Critical FF Hamiltonians have dynamic critical exponent $z \geq 2$.

Proof: From the Gosset–Huang inequality,

$$L^{-\Delta} \lesssim \frac{|\langle \Psi | \mathcal{O}_{\boldsymbol{x}} (\mathbb{1} - G) \mathcal{O}'_{\boldsymbol{y}} | \Psi \rangle|}{\| \mathcal{O}^{\dagger}_{\boldsymbol{x}} | \Psi \rangle \| \| \mathcal{O}'_{\boldsymbol{y}} | \Psi \rangle \|} \leq 2 \exp \left(-\text{const.} \times L \sqrt{\epsilon} \right). \tag{2.5}$$

This inequality breaks for sufficiently large L unless $\epsilon \lesssim 1/L^2$. □

$z \geq 2$ from Gosset–Huang inequality

Critical FF Hamiltonians have dynamic critical exponent $z \geq 2$.

Our argument is highly general because we do not assume

- boundary condition
- spatial dimension
- structure of the lattice
- translational invariance

Also, note that our result can be extended to fermionic FF systems with bosonic local Hamiltonians.

Of course, we should show criticality to use our argument.

Are all gapless FF systems also critical?

$\rightarrow$ No, in general. However, the majority of known gapless FF systems are critical.

Our result: $z \geq 2$ for Markov processes

We also prove $z \geq 2$ for dynamic critical phenomena in certain Markov processes, leaving the contexts of quantum systems.

Critical points	z (numerical)	References
Ising (2D)	$2.1667(5) \geq 2$	Nightingale, Blöte, PRB 62, 1089 (2000).
Ising (3D)	$2.0245(15) \geq 2$	Hasenbusch, PRE 101, 022126 (2020).
Heisenberg (3D)	$2.033(5) \geq 2$	Astillero, Ruiz-Lorenzo, PRE 100, 062117 (2019).
three-state Potts (2D)	$2.193(5) \geq 2$	Murase, Ito, JPSJ 77, 014002 (2008).
four-state Potts (2D)	$2.296(5) \geq 2$	Phys. A: Stat. Mech. Appl. 388, 4379 (2009).

Table 1: Dynamic critical exponents for Markov processes relaxing to critical equilibrium states.

Our framework

Critical FF systems $\Rightarrow z \geq 2$

The essential part of the proof relies on the Gosset–Huang inequality.

When can criticality be shown?

- Rokhsar–Kivelson Hamiltonians (Sec. 3 & 4)
  $\leftarrow$ correspond to Markov processes.
- Plane-wave ground state (Sec. 5)
- Hidden critical correlations of “local” excitations (Sec. 5)

Open question:
Is there a field theoretic explanation? (Sec. 6)

We focus on a specific class of FF Hamiltonians.

Definition 6. RK Hamiltonians

Let

- $\mathcal{S} = \{\mathcal{C}\}$: set of classical configurations (e.g. Ising spins).
- $w(\mathcal{C}) \geq 0$: Boltzmann weight for $\mathcal{C} \in \mathcal{S}$.

$H^{\mathrm{RK}} = \sum_i H_i^{\mathrm{RK}}$ is a Rokhsar–Kivelson (RK) Hamiltonian if

1. Hamiltonian is FF
2. GS can be written as

$$|\Psi_{\mathrm{RK}}\rangle = \sum_{\mathcal{C}\in\mathcal{S}} \sqrt{\frac{w(\mathcal{C})}{\mathcal{Z}}}\,|\mathcal{C}\rangle, \quad \mathcal{Z} = \sum_{\mathcal{C}\in\mathcal{S}} w(\mathcal{C}). \tag{3.1}$$

3. The off-diagonal elements of H_i are non-positive

Note that the properties 2 and 3 are basis dependent.

RK Hamiltonians correspond to Markov processes with local state updates and the detailed balance condition.

Henley, J. Phys.: Condens. Matter 16 S891 (2004).

Castelnovo *et al.*, Ann. Phys. 318, 316 (2005).

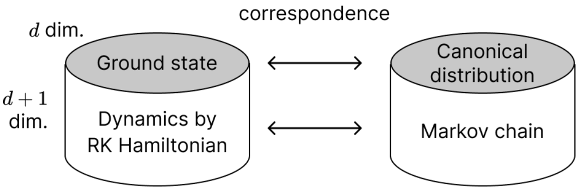


Figure 4: Correspondence between RK Hamiltonians and Markov processes.

Correspondence between RK Hamiltonians and Markov processes

RK Hamiltonians	Markov processes
Hilbert space $\mathcal{H} = \mathrm{Span}\{\lvert\mathcal{C}\rangle\}$	Configuration space $\mathcal{S} = \{\mathcal{C}\}$
Ground state $\sum_{\mathcal{C}\in\mathcal{S}} \sqrt{w(\mathcal{C})/\mathcal{Z}}\,\lvert\mathcal{C}\rangle$	Steady state $w(\mathcal{C})/\mathcal{Z}$
Hamiltonian H^{RK}	Transition-rate matrix W
Symmetricity $(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'} = (H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'}$	Detailed balance condition $(W_i)_{\mathcal{C}\mathcal{C}'} w(\mathcal{C}') = (H_i)_{\mathcal{C}'\mathcal{C}} w(\mathcal{C})$
FF-ness $\langle\Psi_{\mathrm{RK}}\rvert H_i^{\mathrm{RK}} = 0$	Probability conservation $\sum_{\mathcal{C}} (W_i)_{\mathcal{C}\mathcal{C}'} = 0$
Dynamic critical exponent $\mathrm{gap}(H^{\mathrm{RK}}) \sim L^{-z}$	Dynamic critical exponent $\tau \sim L^{z}$

Table 2: Correspondense between RK Hamiltonians and Markov processes

Correspondence between RK Hamiltonians and Markov processes

We define the transition-rate matrix W from the Hamiltonian by

$$W_i = -S H_i^{\mathrm{RK}} S^{-1}, \quad W = \sum_i W_i, \tag{3.2}$$

where

$$S_{\mathcal{C}\mathcal{C}'} = \langle \mathcal{C} | S | \mathcal{C}' \rangle = \sqrt{\frac{w(\mathcal{C})}{\mathcal{Z}}}\, \delta_{\mathcal{C}\mathcal{C}'}. \tag{3.3}$$

Then the imaginary-time Schrödinger equation corresponds to the master equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = -H^{\mathrm{RK}}|\psi\rangle \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t} p(\mathcal{C}) = \sum_{\mathcal{C}' \in \mathcal{S}} W_{\mathcal{C}\mathcal{C}'} p(\mathcal{C}'), \quad p(\mathcal{C}) := \langle \mathcal{C} | S | \psi \rangle. \tag{3.4}$$

The GS $|\Psi_{\mathrm{RK}}\rangle$ corresponds to the steady state $w(\mathcal{C})/\mathcal{Z}$:

$$H^{\mathrm{RK}} |\Psi_{\mathrm{RK}}\rangle = 0 \Leftrightarrow \sum_{\mathcal{C}'} W_{\mathcal{C}\mathcal{C}'} \frac{w(\mathcal{C}')}{\mathcal{Z}} = 0. \tag{3.5}$$

The local Hamiltonian H_i^{RK} is symmetric (Hermitian + real matrix elements). This implies W_i satisfies detailed balance condition:

$$\begin{aligned}(W_i)_{\mathcal{C}\mathcal{C}'}w(\mathcal{C}') &= -\sqrt{w(\mathcal{C})}\,(W_i)_{\mathcal{C}\mathcal{C}'}\frac{1}{\sqrt{w(\mathcal{C}')}}w(\mathcal{C}') \\ &= -\sqrt{w(\mathcal{C})w(\mathcal{C}')}\,(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'} \\ &= -\sqrt{w(\mathcal{C}')w(\mathcal{C})}\,(H_i^{\mathrm{RK}})_{\mathcal{C}'\mathcal{C}} \\ &= (W_i)_{\mathcal{C}'\mathcal{C}}w(\mathcal{C}). \end{aligned} \tag{3.6}$$

Also from $\langle \Psi_{\mathrm{RK}}|H_i^{\mathrm{RK}} = 0$ (FF-ness), we obtain the probability conservation

$$\sum_{\mathcal{C}}(W_i)_{\mathcal{C}\mathcal{C}'} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mathcal{C}\in\mathcal{S}}p(\mathcal{C}) = \sum_{\mathcal{C},\mathcal{C}'\in\mathcal{S}}W_{\mathcal{C}\mathcal{C}'}p(\mathcal{C}') = 0. \tag{3.7}$$

Let us consider the autocorrelation functions

$$A_{\mathcal{O}}(t) := \frac{\langle \Psi_{\mathrm{RK}} | \mathcal{O}(\mathrm{e}^{-H^{\mathrm{RK}}t} - G)\mathcal{O} | \Psi_{\mathrm{RK}} \rangle}{\langle \Psi_{\mathrm{RK}} | \mathcal{O}(\mathbb{1} - G)\mathcal{O} | \Psi_{\mathrm{RK}} \rangle} \tag{3.8}$$

where $G = |\Psi_{\mathrm{RG}}\rangle\langle\Psi_{\mathrm{RG}}|$ is the projector onto ground subspace. (For simplicity, we assume the GS is unique.) The autocorrelation functions satisfy

$$A_{\mathcal{O}}(0) = 1, \quad \lim_{t\to\infty} A_{\mathcal{O}}(t) = 0. \tag{3.9}$$

The decay of the autocorrelation function is characterized by the relaxation time defined as

$$\tau := \frac{1}{\mathrm{gap}(H^{\mathrm{RK}})}. \tag{3.10}$$

If H^{RK} is gapless, τ diverges as $L \to \infty$. Then, the dynamic critical exponent z is defined as $\tau \sim L^z$.

Correspondence between RK Hamiltonians and Markov processes

RK Hamiltonians	Markov processes
Hilbert space $\mathcal{H} = \mathrm{Span}\{\|\mathcal{C}\rangle\}$	Configuration space $\mathcal{S} = \{\mathcal{C}\}$
Ground state $\sum_{\mathcal{C}\in\mathcal{S}} \sqrt{w(\mathcal{C})/\mathcal{Z}}\,\|\mathcal{C}\rangle$	Steady state $w(\mathcal{C})/\mathcal{Z}$
Hamiltonian H^{RK}	Transition-rate matrix W
Symmetricity $(H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'} = (H_i^{\mathrm{RK}})_{\mathcal{C}\mathcal{C}'}$	Detailed balance condition $(W_i)_{\mathcal{C}\mathcal{C}'} w(\mathcal{C}') = (W_i)_{\mathcal{C}'\mathcal{C}} w(\mathcal{C})$
FF-ness $\langle \Psi_{\mathrm{RK}}\|H_i^{\mathrm{RK}} = 0$	Probability conservation $\sum_{\mathcal{C}} (W_i)_{\mathcal{C}\mathcal{C}'} = 0$
Dynamic critical exponent $\mathrm{gap}(H^{\mathrm{RK}}) \sim L^{-z}$	Dynamic critical exponent $\tau \sim L^{z}$

Table 3: Correspondense between RK Hamiltonians and Markov processes

First, we roughly introduce a numerical way to compute z. We discretize the Markov process by

$$\mathrm{e}^{Wt} \approx (1 + W\delta t)^{t/\delta t}, \tag{4.1}$$

$$\mathrm{e}^{-Ht} \approx (\mathbb{1} - H\delta t)^{t/\delta t} = \exp\left(-\frac{\ln(\mathbb{1} - H\delta t)}{-\delta t} t\right). \tag{4.2}$$

The continuous and discrete dynamics share the same dynamic critical exponent. The discretized Markov process can be simulated by Markov chain Monte Carlo (MCMC) methods.

$$\mathcal{C}(0) \xrightarrow{\mathbb{1}+W\delta t} \mathcal{C}(\delta t) \xrightarrow{\mathbb{1}+W\delta t} \mathcal{C}(2\delta t) \xrightarrow{\mathbb{1}+W\delta t} \cdots \xrightarrow{\mathbb{1}+W\delta t} C(t). \tag{4.3}$$

We can compute the dynamic critical exponents z numerically by measuring relaxations of autocorrelation functions (in a much shorter time than for exact diagonalization).

Example: 2+1D kinetic Ising model

■ Gibbs sampling for 2D critical Ising model

Let Λ be the square lattice and let $\mathcal{C} = \{\sigma_i\}_{i\in\Lambda}$. Each spin σ_i takes the values of ± 1. The Boltzmann weight of the Ising model is given by

$$w(\mathcal{C}) = \mathrm{e}^{-\beta E(\mathcal{C})}, \quad E(\mathcal{C}) = -\sum_{\langle i,j\rangle} \sigma_i \sigma_j. \tag{4.4}$$

Let $\mathcal{C}_i$ be the configuration obtained by flipping the spin at $i \in \Lambda$ in $\mathcal{C}$. The local transition rate matrix of the Gibbs sampling is given by

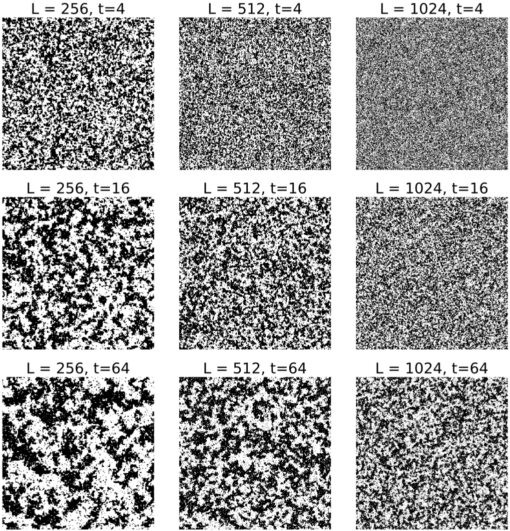
$$(W_i)_{\mathcal{C}_i\mathcal{C}} = -(W_i)_{\mathcal{C}\mathcal{C}} = \frac{w(\mathcal{C}_i)}{w(\mathcal{C}) + w(\mathcal{C}_i)}. \tag{4.5}$$

Corresponding RK Hamiltonian is

$$H_i^{\mathrm{RK}} = \frac{1}{2\cosh(\beta \sum_{j\sim i} Z_j)} \left(\mathrm{e}^{-\beta Z_i \sum_{j\sim i} Z_j} - X_i\right). \tag{4.6}$$

Example: 2+1D kinetic Ising model

At $\beta = \beta_c = \frac{1}{2}\ln(1+\sqrt{2})$, the relaxation time diverges as $L \to \infty$. ($z \approx 2.17$)



Dynamic critical exponents for various critical points

Critical points	z (numerical)	References
Ising (2D)	$2.1667(5) \geq 2$	Nightingale, Blöte, PRB 62, 1089 (2000).
Ising (3D)	$2.0245(15) \geq 2$	Hasenbusch, PRE 101, 022126 (2020).
Heisenberg (3D)	$2.033(5) \geq 2$	Astillero, Ruiz-Lorenzo, PRE 100, 062117 (2019).
three-state Potts (2D)	$2.193(5) \geq 2$	Murase, Ito, JPSJ 77, 014002 (2008).
four-state Potts (2D)	$2.296(5) \geq 2$	Phys. A: Stat. Mech. Appl. 388, 4379 (2009).

Table 4: Dynamic critical exponents of RK Hamiltonians corresponding to critical statistical systems.

RK Hamiltonians constructed from the Boltzmann weight of a critical point seemed to have dynamic critical exponent $z \geq 2$.

$\leftarrow$ Conjectured by Isakov *et al.* PRB 83, 125114 (2011).

Critical FF systems

Let us show $z \geq 2$ for RK Hamiltonians constructed from critical statistical systems.

We recap the definition of criticality for FF systems and its implications. An FF Hamiltonian is critical if there is a correlation function such that

$$|\boldsymbol{x}-\boldsymbol{y}| \sim L, \quad \frac{|\langle\Psi|\mathcal{O}_{\boldsymbol{x}}(\mathbb{1}-G)\mathcal{O}'_{\boldsymbol{y}}|\Psi\rangle|}{\|\mathcal{O}^{\dagger}_{\boldsymbol{x}}|\Psi\rangle\|\|\mathcal{O}'_{\boldsymbol{y}}|\Psi\rangle\|} \gtrsim L^{-\Delta}, \quad \Delta > 0. \tag{4.7}$$

Critical FF Hamiltonians satisfy $z \geq 2$.

Theorem 2. Masaoka, Soejima, Watanabe arXiv:2406.06415.

The RK Hamiltonian with a unique GS constructed from the Boltzmann weight of a critical point is a critical FF system and its dynamic critical exponent satisfies $z \geq 2$.

Let us show the criticality of this model. For diagonal operators $O := \sum_{\mathcal{C}\in\mathcal{S}} O(\mathcal{C})|\mathcal{C}\rangle\langle\mathcal{C}|$, quantum expectations corresponds to classical expectations:

$$\langle\Psi_{\mathrm{RK}}|O|\Psi_{\mathrm{RK}}\rangle = \sum_{\mathcal{C}\in\mathcal{S}} \frac{O(\mathcal{C})w(\mathcal{C})}{\mathcal{Z}} =: \langle O\rangle. \tag{4.8}$$

Since the Boltzmann weight $w(\mathcal{C})$ is at a critical point, there is a local operator O_i such that

$$\langle O_i\rangle = 0, \quad \langle O_i^2\rangle = \text{const.}, \quad \langle O_i O_j\rangle \sim \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^{2\Delta_O}}, \tag{4.9}$$

where Δ_O is the scaling dimension of O_i. Thus, if $|\boldsymbol{x}_i - \boldsymbol{x}_j| \sim L$,

$$\begin{aligned}\frac{|\langle\Psi_{\mathrm{RK}}|\mathcal{O}_i(\mathbb{1}-G)\mathcal{O}_j|\Psi_{\mathrm{RK}}\rangle|}{\|\mathcal{O}_i|\Psi_{\mathrm{RK}}\rangle\|\|\mathcal{O}_j|\Psi_{\mathrm{RK}}\rangle\|} &\sim |\langle\Psi_{\mathrm{RK}}|\mathcal{O}_i(1-|\Psi_{\mathrm{RK}}\rangle\langle\Psi_{\mathrm{RK}}|)\mathcal{O}_j|\Psi_{\mathrm{RK}}\rangle| \\ &= |\langle\mathcal{O}_i\mathcal{O}_j\rangle - \langle\mathcal{O}_i\rangle\langle\mathcal{O}_j\rangle| \sim L^{-2\Delta_{\mathcal{O}}}.\end{aligned} \tag{4.10}$$

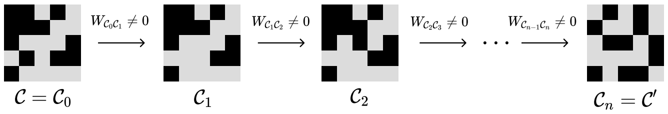
Here, we used $G = |\Psi_{\mathrm{RK}}\rangle\langle\Psi_{\mathrm{RK}}|$ since the GS is unique. Therefore, this model is critical, and $z \geq 2$ from our theorem.

We used the uniqueness of the GS. This assumption is justified by ergodicity.

Definition 7. Ergodicity

A Markov process with transition-rate W is called ergodic if, $\forall(\mathcal{C},\mathcal{C}')$, there exist $n \in \mathbb{N}$ and a chain of configurations $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$ such that

$$\mathcal{C}_0 = \mathcal{C}, \quad \mathcal{C}_n = \mathcal{C}', \quad W_{\mathcal{C}_0\mathcal{C}_1} W_{\mathcal{C}_1\mathcal{C}_2} \cdots W_{\mathcal{C}_{n-1}\mathcal{C}_n} \neq 0. \qquad (4.11)$$



An ergodic Markov process has a unique steady state. The proof is based on the Perron–Frobenius theorem.

Non-ergodic Markov processes have completely separated configuration subspaces. In this case, we focus on one of them to recover ergodicity.

No-go theorem for local MCMC methods with detailed balance

For RK Hamiltonians constructed by critical points, we can show criticality by the same argument. Thus, the following no-go theorem follows.

No-go theorem

Ergodic Markov processes with local state updates and the detailed balance condition undergo critical slowing down at a critical point, with a dynamic critical exponent $z \geq 2$.

$\rightarrow$ First proof of an empirical fact known in the MCMC contexts.

Remark.

We can consider FF Hamiltonians with more general ground states that have a phase factor:

$$|\Psi\rangle = \sum_{\mathcal{C}\in\mathcal{S}} \mathrm{e}^{\mathrm{i}\theta(\mathcal{C})} \sqrt{\frac{w(\mathcal{C})}{\mathcal{Z}}}\, |\mathcal{C}\rangle, \quad \theta(\mathcal{C}) \in \mathbb{R}. \tag{4.12}$$

■ Fine-tuned Fibonacci Levin Wen model

Fendley, Fradkin, PRB 72, 024412 (2005).

Fendley, Ann. Phys. 323(12), 3113-3136 (2008).

- The Boltzmann weight $w(\mathcal{C})$ represents $c = 14/15$ CFT.
- Ground state shows algebraic correlations.
- It cannot be mapped to MCMC due to the sign problem.

We can show $z \geq 2$ in the same way since phases $\pm\theta(\mathcal{C})$ cancel in correlation functions of diagonal operators.

By violating the assumptions in the no-go theorem, one can create Markov processes with faster relaxation with $z < 2$.

- Wolff cluster algorithm Wolff, PRL. 62, 361 (1988).

Locality: $\times$, Detailed balance condition: $\checkmark$

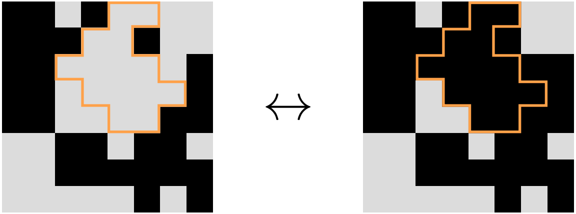


Figure 5: State update of the Wolff cluster algorithm

The dynamic critical exponent is $z \approx 0.3$ for the 2D Ising critical point.

Liu *et al.* PRB 89, 054307 (2014).

■ Asymmetric simple exclusion process (ASEP)

Locality: ✓, Detailed balance condition: ×

Let us consider the following XXZ model with a non-Hermitian term.

$$H_i = \frac{1}{4}(1 - \Delta Z_i Z_{i+1}) - \frac{1+s}{2}\sigma_i^+ \sigma_{i+1}^- - \frac{1-s}{2}\sigma_i^- \sigma_{i+1}^+ + \frac{s}{2}(Z_i - Z_{i+1}) \quad (4.13)$$

$\Delta < 1$: Gapless phase ($z = 1$)

$\Delta > 1$: Gapped phase

$\Delta = 1$: Stochastic line

- $s = 0$: Heisenberg model ($z = 2$)
- $s > 0$: ASEP ($z = 3/2$)

Kim, PRE 52, 3512 (1995).

Gwa, Spohn, PRA 46, 844 (1992).

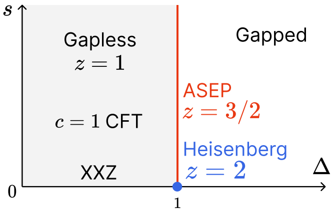


Figure 6: Phase diagram of XXZ model with a non-Hermitian term.

Let us look at examples of critical FF models that are not constructed from critical statistical systems.

- XXZ model with fine-tuned magnetic field

Let us consider the following XXZ model

$$H_i = -X_i X_{i+1} - Y_i Y_{i+1} - \Delta Z_i Z_{i+1} \\ - h(Z_i + Z_{i+1}) + (1+h)\mathbb{1}. \qquad (5.1)$$

We assume the model is on the critical line

$$h + \Delta = 1, \quad \Delta < 1. \qquad (5.2)$$

Then, the kernel of H_i is given by

$$\ker H_i = \mathrm{Span}\{|00\rangle, |01\rangle + |10\rangle\}. \qquad (5.3)$$

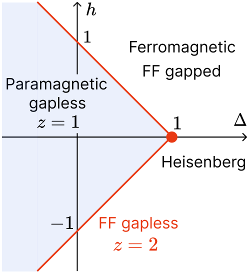


Figure 7: The model (5.1) correpond to upper critical line.

This model is obtained from the zero-temperature limit of the following RK Hamiltonian that conserves particle parity.

$$w(\{n\}) = \mathrm{e}^{-\beta \sum_i n_i}, \tag{5.4}$$

$$\ker H_i = \mathrm{Span}\{|00\rangle + \mathrm{e}^{-\beta}|11\rangle, |01\rangle + |10\rangle\}. \tag{5.5}$$

Figure 8: Phase diagram for β

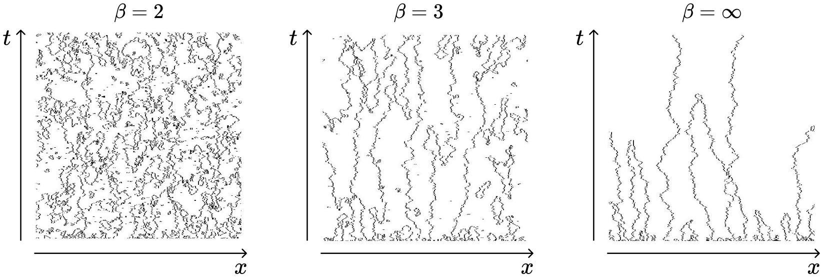


Figure 9: MCMC simulations for various β. The length of the system is $L = 256$ and a periodic boundary condition is imposed. At $\beta = \infty$, relaxation time diverges as $L \to \infty$.

FF systems with plane-wave ground states

Let us show the criticality of this model.

$$H_i = -X_i X_{i+1} - Y_i Y_{i+1} - \Delta Z_i Z_{i+1} - (1-\Delta)(Z_i + Z_{i+1}) + (2-\Delta)\mathbb{1}, \quad (5.6)$$

$$\ker H_i = \mathrm{Span}\{|00\rangle, |01\rangle + |10\rangle\}. \quad (5.7)$$

There are the following two ground states.

$$|\Psi_0\rangle = |0 \cdots 0\rangle,\ |\Psi_1\rangle = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \sigma_i^- |\Psi_0\rangle$$

The second plane-wave ground state is important. We consider the following correlation function.

$$|\langle \Psi_0 | \sigma_i^+ (\mathbb{1} - G) \sigma_j^- |\Psi_0\rangle| = |\langle \Psi_0 | \sigma_i^+ |\Psi_1\rangle \langle \Psi_1 | \sigma_j^- |\Psi_0\rangle| = \frac{1}{L}. \quad (5.8)$$

Therefore, this model is critical and $z \geq 2$. Note that our definition of criticality does not require $\langle \Psi_0 | a_i (\mathbb{1} - G) a_j^\dagger |\Psi_0\rangle \sim 1/|i-j|^\Delta$.

If an FF system has a plane-wave ground state, then $z \geq 2$ can be said by the same argument.

■ 1+1D zero-temperature kinetic Ising model

The 1+1D kinetic Ising model is the RK Hamiltonian for the 1D Ising model. The Hamiltonian in the zero-temperature limit is

$$H_i = \frac{1}{2}\mathbb{1} - \frac{1}{4}(Z_{i-1}Z_i + Z_iZ_{i+1} + X_i - Z_{i-1}X_iZ_{i+1}) \geq 0. \tag{5.9}$$

$$\ker H_i = \mathrm{Span}\{|000\rangle, |111\rangle, |0{+}1\rangle, |1{+}0\rangle\}. \tag{5.10}$$

The ground states for PBC are

$$|\Psi_0\rangle = |0\cdots0\rangle, \quad |\Psi_1\rangle = |1\cdots1\rangle. \tag{5.11}$$

These states do not have any correlation at first glance. However, this model has the dynamic critical exponent $z = 2$.

How to detect criticality? We define the following "local" excitations.

$$|O_i^+\rangle := |\overbrace{0\cdots0}^{i}10\cdots0\rangle + |\overbrace{0\cdots0}^{i}110\cdots0\rangle + \cdots + |\overbrace{1\cdots10}^{i}1\cdots1\rangle, \tag{5.12}$$

$$|O_i^-\rangle := |\overbrace{1\cdots1}^{i}01\cdots1\rangle + |\overbrace{1\cdots1}^{i}001\cdots1\rangle + \cdots + |\overbrace{0\cdots01}^{i}0\cdots0\rangle. \tag{5.13}$$

These states can be treated as local excitations since

$$H_j|O_i^+\rangle = H_j|O_i^-\rangle = 0 \quad (j \neq i, i+1). \tag{5.14}$$

Our argument works for such extended cases as well. The correlation function for O_i^+ and O_j^- is

$$\frac{|\langle O_i^+|(\mathbb{1} - G)|O_j^-\rangle|}{\||O_i^+\rangle\|\||O_j^-\rangle\|} = \cdots = \frac{1}{L-1}. \tag{5.15}$$

Thus, this model is critical and $z \geq 2$.

Definition 8. FF field teory

A field theory is frustration-free if the Hamiltonian density $\mathcal{H}(x)$ of the model is positive semidefinite, and a ground state $|\Psi\rangle$ satisfies

$$\forall x,\ \mathcal{H}(x)|\Psi\rangle = 0, \tag{6.1}$$

where x is the spatial coordinate.

In the following few slides, let us look at some examples of FF field theory.

An important example of FF field theories is the non-interacting Schrödinger field theory, whose action is given by

$$S[\psi, \psi^\dagger] = \int \mathrm{d}^d x \, \mathrm{d}t \, \psi^\dagger(x,t) \left(\mathrm{i}\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi(x,t). \tag{6.2}$$

The Hamiltonian density is

$$\mathcal{H}(x) = \frac{1}{2m} \nabla\psi^\dagger(x) \cdot \nabla\psi(x), \tag{6.3}$$

where $\psi^\dagger$ and ψ are creation and annihilation operators that satisfy $[\psi(x), \psi^\dagger(y)] = \delta(x-y)$. The ground state is the Fock vacuum $|0\rangle$ and

$$\mathcal{H}(x)|0\rangle = \frac{1}{2m} \nabla\psi^\dagger(x) \cdot \nabla\psi(x)|0\rangle = 0. \tag{6.4}$$

Thus this model is FF.
This model is exactly solvable and the dispersion relation is $\epsilon = k^2/2m$, where k is the wavenumber. We obtain $z = 2$ by taking $k = 2\pi/L$.

We can construct the $d+1$-dimensional FF field theory from the action of a d-dimensional field theory by stochastic quantization, which is the field-theoretic counterpart of the RK Hamiltonians.

Parisi, Wu, Sci. sin, 24(4), 483-496, (1981). Dijkgraaf, Orlando, Reffert, arxiv:0903.0732 (2009)

Let us consider a stochastic dynamics given by the following master equation (Fokker–Planck equation).

$$\frac{\partial}{\partial t} P[\phi, t] = W P[\phi, t] = \frac{1}{2} \int \mathrm{d}^d x \, \frac{\delta}{\delta \phi(x)} \left(\frac{\delta S_{\mathrm{cl}}}{\delta \phi(x)} + \frac{\delta}{\delta \phi(x)} \right) P[\phi, t], \tag{6.5}$$

where $P[\phi, t]$ is a probability distribution and $S_{\mathrm{cl}}[\phi]$ is the action of a d-dimensional Euclidean field theory. The steady state is given by

$$P^{\mathrm{eq}}[\phi] = \frac{\mathrm{e}^{-S_{\mathrm{cl}}[\phi]}}{\mathcal{Z}_{\mathrm{cl}}}, \quad \mathcal{Z}_{\mathrm{cl}} := \int \mathcal{D}\phi \, \mathrm{e}^{-S_{\mathrm{cl}}[\phi]} \tag{6.6}$$

We define the wave functional by

$$\psi[\phi,t] := \sqrt{\frac{\mathcal{Z}_{\mathrm{cl}}}{\mathrm{e}^{-S_{\mathrm{cl}}[\phi]}}}\,P[\phi,t] \tag{6.7}$$

One can obtain the following imaginary time Schrödinger equation from the master equation.

$$\frac{\partial}{\partial t}\psi[\phi,t] = -H\psi[\phi,t] = -\int \mathrm{d}^d x\,\mathcal{H}(x)\psi[\phi,t], \tag{6.8}$$

where

$$\mathcal{H}(x) = \frac{1}{2}\mathcal{Q}(x)^\dagger\mathcal{Q}(x), \quad \mathcal{Q}(x) := \frac{\delta}{\delta\phi(x)} + \frac{1}{2}\frac{\delta S_{\mathrm{cl}}}{\delta\phi(x)}. \tag{6.9}$$

This Hamiltonian density is positive semidefinite. THe GS is given by

$$\Psi[\phi] = \sqrt{\frac{\mathcal{Z}_{\mathrm{cl}}}{\mathrm{e}^{-S_{\mathrm{cl}}[\phi]}}}\,P^{\mathrm{eq}}[\phi] = \sqrt{\frac{\mathrm{e}^{-S_{\mathrm{cl}}[\phi]}}{\mathcal{Z}_{\mathrm{cl}}}} \tag{6.10}$$

This model is FF since $\mathcal{H}(x)\Psi[\phi] = \frac{1}{2}\mathcal{Q}(x)^\dagger\mathcal{Q}(x)\Psi[\phi] = 0$.

For example, let

$$S_{\mathrm{cl}}[\phi] = \kappa \int \mathrm{d}^d x \, (\nabla \phi(x))^2. \tag{6.11}$$

The stochastic quantization yields the following Hamiltonian.

$$\mathcal{H}(x) = -\frac{1}{2} \frac{\delta^2}{\delta \phi(x)^2} + \frac{\kappa^2}{2} (\nabla^2 \phi(x))^2 + \mathrm{const}. \tag{6.12}$$

Corresponding $d+1$-dimensional action is

$$S[\phi] = \int \mathrm{d}^d x \, \mathrm{d}t \left(\frac{1}{2} (\partial_0 \phi(x,t))^2 + \frac{\kappa^2}{2} (\nabla^2 \phi(x,t))^2 \right). \tag{6.13}$$

This model is called the quantum Lifshitz theory. This model is solvable and we obtain $z = 2$.

The $d+1$-dimensional field theory obtained by stochastic quantization of a d-dimensional CFT can be considered an effective theory of the RK Hamiltonian of a critical point. Our results provide microscopic explanation of $z \geq 2$ in this case. However, field theoretic understanding is still lacking.

General conjecture on FF field theories

The dynamic critical exponents of gapless FF field theories satisfy $z \geq 2$.

1. Introduction
2. Previous studies and our results
3. Rokhsar–Kivelson Hamiltonians and Markov processes
4. Critical FF systems from critical points
5. Other critical FF systems
6. FF field theory
7. Summary and open questions

Supplements

Summary

Our study highlights the unique nature of the gapless FF system through the dynamic critical exponent. We have established the lower bound $z \geq 2$ for critical FF systems. This class contains

- RK Hamiltonians constructed from critical points
- FF systems with a plane-wave ground state.
- FF systems with a hidden long-range correlation.

Also, we established the following no-go theorem for Markov processes.

- Local Markov processes with the detailed balance condition undergo critical slowing down at a critical point with $z \geq 2$.

Surprisingly, new insights can be gained in the traditional field of dynamic critical phenomena by employing knowledge from quantum theory.

Is there a general proof of $z \geq 2$ for gapless FF systems?

We assumed the existence of a critical correlation function. Is there always a critical correlation function in a gapless FF system?

How fast does non-Hermiticity (breaking detailed balance) speed up relaxation?

Is there a field-theoretic proof or understanding of why $z \geq 2$?

THANK YOU.

1. Introduction

2. Previous studies and our results

3. Rokhsar–Kivelson Hamiltonians and Markov processes

4. Critical FF systems from critical points

5. Other critical FF systems

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Supplements

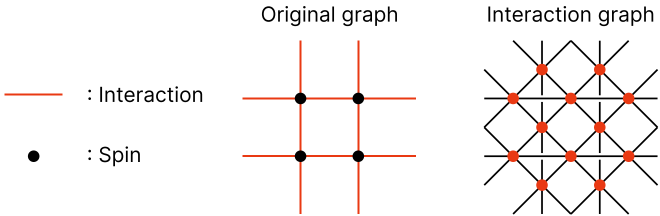
Let us review the derivation of the Gosset–Huang inequality.

Gosset, Huang, PRL 116, 097202 (2016).

Definition 9. Interaction graph

- Vertices: $1, \ldots, N =$ number of H_i.
- i and j are adjacent $(i \sim j)$ if $[H_i, H_j] \neq 0$.
- g_i: degree of $i =$ number of vertices adjacent to i.
- $g := \max_i g_i$.

■ Nearest neighbor interactions on the square lattice $(g = 6)$



Definition 10. Distance between local Hamiltonians

Distance $\tilde{d}(H_i, H_j)$ between H_i and H_j is given by the number of edges in the shortest path connecting i and j.

Definition 11. Distance between operators

$$\tilde{d}(\mathcal{O}, \mathcal{O}') := 2 + \min\{\tilde{d}(H_i, H_j) \mid [\mathcal{O}, H_i] \neq 0, [\mathcal{O}', H_j] \neq 0\} \tag{8.1}$$

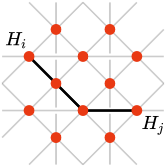


Figure 10: $\tilde{d}(H_i, H_j) = 3$

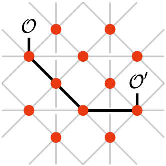


Figure 11: $\tilde{d}(\mathcal{O}, \mathcal{O}') = 5$

Definition 12. Chromatic number

The chromatic number c is the smallest number of colors needed for the coloring $i \mapsto \mathrm{color}(i) \in \{1, \dots, c\}$ such that

$$i \sim j \Rightarrow \mathrm{color}(i) \neq \mathrm{color}(j). \tag{8.2}$$

- $c = 2$ for bipartite graphs
- $c \leq 4$ for planar graphs (four-color theorem)
- The greedy algorithm ensures $c \leq g + 1$.

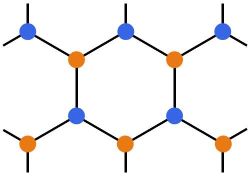


Figure 12: Coloring of the honeycomb lattice

First, we replace each local Hamiltonian with a projector while preserving its kernel. This operation does not change the ground states and the dynamic critical exponent.

Theorem 3. Gosset–Huang inequality Gosset, Huang, PRL 116, 097202 (2016).

Let

- $|\Psi\rangle$: GS of H
- G : Projector onto the ground subspace
- c, g : Chromatic number and maximum degree
- ϵ : $\mathrm{gap}(H)$

Then

$$\frac{|\langle\Psi|\mathcal{O}(\mathbb{1}-G)\mathcal{O}'|\Psi\rangle|}{\|\mathcal{O}^\dagger|\Psi\rangle\|\|\mathcal{O}'|\Psi\rangle\|} \leq 2\exp\left(-\frac{2(\tilde{d}(\mathcal{O},\mathcal{O}')-2)}{2c-1}\sqrt{\frac{\epsilon}{g^2+\epsilon}}\right). \quad (8.3)$$

Definition 13. Operator norm

$\|A\| := \max_{|\psi\rangle \neq 0} \|A|\psi\rangle\| / \||\psi\rangle\|$.

Lemma 1. Detectability lemma Anshu, Arad, Vidick, PRB 93, 205142 (2016).

We assume

- $H = \sum_{i=1}^{N} H_i$ is FF,
- Each H_i is an orthogonal projector.

Let $P_i := \mathbb{1} - H_i$ and $P := P_{\sigma(1)} \cdots P_{\sigma(N)}$ for arbitrary permutation $\sigma \in S_N$. Then

$$\|P - G\| \leq \sqrt{\frac{g^2}{g^2 + \epsilon}} = 1 - O(\epsilon), \quad \epsilon = \mathrm{gap}(H), \tag{8.4}$$

where G is the projector to the ground subspace of H, and g is the maximum degree of the interaction graph for $\{H_i\}$.

Proof of the detectability lemma

Proof. Let us consider the quantity $\|H_i P_{\sigma(j)} \cdots P_{\sigma(N)} |\psi\rangle\|$ for any state $|\psi\rangle$ and perform the following operations. If H_i commute with $H_{\sigma(j)}$, we use

$$\begin{aligned}\|H_i P_{\sigma(j)} \cdots P_{\sigma(N)} |\psi\rangle\| &= \|P_{\sigma(j)} H_i P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\| \\ &\leq \|H_i P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\|. \end{aligned} \tag{8.5}$$

Otherwise, we use

$$\begin{aligned}&\|H_i P_{\sigma(j)} P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\| \\ &\leq \|H_i P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\| + \|H_i H_{\sigma(j)} P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\| \\ &\leq \|H_i P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\| + \|H_{\sigma(j)} P_{\sigma(j+1)} \cdots P_{\sigma(N)} |\psi\rangle\|. \end{aligned} \tag{8.6}$$

If $\sigma(j) = i$, this procedure stops since $H_i P_i = 0$. If we start from $\|H_i P |\psi\rangle\|$ we obtain the sum of at most g_i terms:

$$\|H_i P |\psi\rangle\| \leq \sum_{\sigma(l) \sim i} \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)} |\psi\rangle\|. \tag{8.7}$$

Proof of the detectability lemma

Since the square of the average can be bounded above by the average of the square, we obtain

$$\begin{aligned}
\|H_i P|\psi\rangle\|^2 &\le g_i^2 \left(\frac{1}{g_i} \sum_{\sigma(l)\sim i} \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)}|\psi\rangle\| \right)^2 \\
&\le g_i^2 \frac{1}{g_i} \sum_{\sigma(l)\sim i} \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)}|\psi\rangle\|^2. \qquad (8.8)
\end{aligned}$$

Summing the left-hand side from $i = 1$ to N yields the energy expectation for $P|\psi\rangle$. Since there are at most g vertices H_i adjacent to $H_{\sigma(l)}$,

$$\begin{aligned}
\langle\psi|P^\dagger H P|\psi\rangle = \sum_{i=1}^{N} \|H_i P|\psi\rangle\|^2 &\le \sum_{i=1}^{N} g_i \sum_{l:\,\sigma(l)\sim i} \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)}|\psi\rangle\|^2 \\
&\le \sum_{l=1}^{N} \sum_{i:\, i\sim\sigma(l)} g_i \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)}|\psi\rangle\|^2 \\
&\le g^2 \sum_{l=1}^{N} \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)}|\psi\rangle\|^2. \qquad (8.9)
\end{aligned}$$

Proof of the detectability lemma

$$\begin{aligned}
\text{R.H.S of (8.9)} &= g^2 \sum_{l=1}^{N} \|H_{\sigma(l)} P_{\sigma(l+1)} \cdots P_{\sigma(N)} |\psi\rangle\|^2 \\
&= g^2 \sum_{l=1}^{N} \langle\psi| P_{\sigma(N)} \cdots P_{\sigma(l+1)} (\mathbb{1} - P_{\sigma(l)}) P_{\sigma(l+1)} \cdots P_{\sigma(N)} |\psi\rangle \\
&= g^2 (\||\psi\rangle\|^2 - \|P|\psi\rangle\|^2). \qquad (8.10)
\end{aligned}$$

Let $|\psi^\perp\rangle = (\mathbb{1} - G)|\psi\rangle$. The state $P|\psi^\perp\rangle$ is orthogonal to the ground states since $GP|\psi^\perp\rangle = G|\psi^\perp\rangle = 0$. Therefore

$$\epsilon \|P|\psi^\perp\rangle\|^2 \leq \langle\psi^\perp| P^\dagger H P |\psi^\perp\rangle \leq g^2 (\||\psi^\perp\rangle\|^2 - \|P|\psi^\perp\rangle\|^2), \qquad (8.11)$$

where ϵ is the spectral gap of H. Thus

$$\|P - G\| = \max_{|\psi\rangle \neq 0} \frac{\|(P - G)|\psi\rangle\|}{\||\psi\rangle\|} = \max_{|\psi\rangle \neq 0} \frac{\|P|\psi^\perp\rangle\|}{\||\psi\rangle\|} \leq \sqrt{\frac{g^2}{g^2 + \epsilon}}. \qquad (8.12)$$

$\square$

Lemma 2.

We divide $\{H_i\}$ into c colors so that no two adjacent vertices have the same color. We denote i-th local Hamiltonian with color j as $H_i^{(j)}$. Let $P_i^{(j)} := \mathbb{1} - H_i^{(j)}$, and consider

$$P := \prod_i P_i^{(c)} \prod_i P_i^{(c-1)} \cdots \prod_i P_i^{(2)} \prod_i P_i^{(1)}. \tag{8.13}$$

Then

$$\langle \Psi | \mathcal{O}\mathcal{O}' | \Psi \rangle = \langle \Psi | \mathcal{O} (P^\dagger P)^n \mathcal{O}' | \Psi \rangle \quad \text{for} \quad n \le m, \tag{8.14}$$

where

$$m := \frac{\tilde{d}(\mathcal{O}, \mathcal{O}') - 2}{2c - 1}. \tag{8.15}$$

Proof: Since $|\Psi\rangle$ is the ground state, $\langle\Psi|\mathcal{O}P_i^{(1)} = \langle\Psi|\mathcal{O}(\mathbb{1} - H_i^{(1)}) = \langle\Psi|\mathcal{O}$ if $[H_i^{(1)}, \mathcal{O}] = 0$. Repeating this argument,

$$\begin{aligned}\langle\Psi|\mathcal{O}P^\dagger P &= \langle\Psi|\mathcal{O}\prod_i P_i^{(1)}\prod_i P_i^{(2)}\cdots\prod_i P_i^{(c)}\prod_i P_i^{(c-1)}\cdots\prod_i P_i^{(2)}\prod_i P_i^{(1)} \\ &= \langle\Psi|\mathcal{O}\prod_i(\mathbb{1} - H_i^{(1)})\cdots\prod_i(\mathbb{1} - H_i^{(c)})\cdots\prod_i(\mathbb{1} - H_i^{(1)}) \\ &= \langle\Psi|\mathcal{O}\prod_{i:\,[H_i^{(1)},\mathcal{O}]\neq 0} P_i^{(1)}\prod_{i:\,\tilde{d}(H_i^{(2)},\mathcal{O})\leq 2} P_i^{(2)}\prod_{i:\,\tilde{d}(H_i^{(3)},\mathcal{O})\leq 3} P_i^{(3)}\cdots\prod_{i:\,\tilde{d}(H_i^{(1)},\mathcal{O})\leq 2c-1} P_i^{(1)}.\end{aligned} \tag{8.16}$$

Thus, only $P_i^{(j)}$ in the "light cone" remain. Therefore,

$$\langle\Psi|\mathcal{O}\mathcal{O}'|\Psi\rangle = \langle\Psi|\mathcal{O}(P^\dagger P)^n\mathcal{O}'|\Psi\rangle$$

as long as two light cones from $\mathcal{O}$ and $\mathcal{O}'$ do not overlap (Fig. 13). Since $P^\dagger P$ has $2c-1$ colors, $n \leq m := (\tilde{d}(\mathcal{O}, \mathcal{O}') - 2)/(2c-1)$.

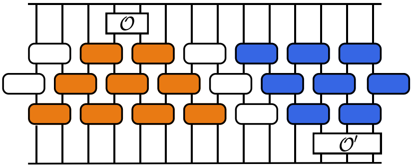


Figure 13: light cones for 1D chain

Proof of Gosset–Huang inequality

We assume that all local Hamiltonians are orthogonal projectors. Let

- $|\Psi\rangle$: GS of H
- G : Projector onto the ground subspace
- c, g : Chromatic number and maximum degree
- ϵ : $\mathrm{gap}(H)$

Then

$$\frac{|\langle\Psi|\mathcal{O}(\mathbb{1}-G)\mathcal{O}'|\Psi\rangle|}{\|\mathcal{O}^\dagger|\Psi\rangle\|\|\mathcal{O}'|\Psi\rangle\|} \leq 2\exp\left(-\frac{2(\tilde{d}(\mathcal{O},\mathcal{O}')-2)}{2c-1}\sqrt{\frac{\epsilon}{g^2+\epsilon}}\right). \quad (8.17)$$

Proof of Gosset–Huang inequality

Let us show the Gosset–Huang inequality. From the lemma 2,

$$\langle\Psi|\mathcal{O}\mathcal{O}'|\Psi\rangle = \langle\Psi|\mathcal{O}Q_m(P^\dagger P)\mathcal{O}'|\Psi\rangle. \tag{8.18}$$

for polynomials $Q_m(x)$ such that $\deg Q_m(x) \leq m$ and $Q_m(1) = 1$. Let $G^\perp := \mathbb{1} - G$. Since $P^\dagger PG = G$,

$$(P^\dagger P)^n - G = (P^\dagger P)^n G^\perp = (P^\dagger P - G)^n G^\perp \tag{8.19}$$

Therefore,

$$\begin{aligned}\langle\Psi|\mathcal{O}(\mathbb{1} - G)\mathcal{O}'|\Psi\rangle &= \langle\Psi|\mathcal{O}(Q_m(P^\dagger P) - G)\mathcal{O}'|\rangle \\ &= \langle\Psi|\mathcal{O}Q_m(P^\dagger P - G)G^\perp\mathcal{O}'|\Psi\rangle \\ &\leq \|\mathcal{O}^\dagger|\Psi\rangle\|\|\mathcal{O}'|\Psi\rangle\|\|Q_m(P^\dagger P - G)\|. \end{aligned} \tag{8.20}$$

We can obtain an upper bound for correlation functions from the upper bound for $\|Q_m(P^\dagger P - G)\|$.

Proof of Gosset–Huang inequality

From the detectability lemma, $\|P^\dagger P - G\| = \|P - G\|^2 \le g^2/(g^2+\epsilon) =: 1-\delta$. Therefore

$$\|Q_m(P^\dagger P - G)\| \le \max_{0\le x\le 1-\delta} |Q_m(x)| \quad \text{where} \quad \delta := \frac{\epsilon}{g^2+\epsilon}. \tag{8.21}$$

We minimize the right-hand side of Eq. (8.21) under the constraint $\deg Q_m \le m$ and $Q_m(1) = 1$. The optimal polynomial is

$$Q_m(x) = \frac{T_m(\frac{2x}{1-\delta} - 1)}{T_m(\frac{2}{1-\delta} - 1)}, \tag{8.22}$$

where $T_m(x)$ is the degree m Chebyshev polynomial of the first kind defined by $T_m(x) = \cos(m \arccos x)$ or $\cosh(m \operatorname{arccosh} x)$.

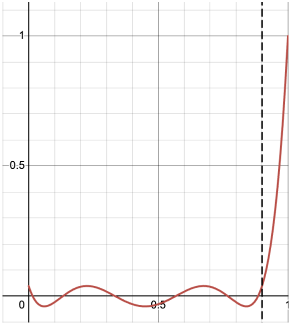


Figure 14: The optimal polynomial $Q_m(x)$ for $m = 6$ and $\delta = 0.1$

The Chebyshev polynomial $T_m(x)$ satisfies

$$\begin{cases} T_m(x) \le \frac{1}{2}\mathrm{e}^{2m\sqrt{(x-1)/(x+1)}} & x \ge 1, \\ |T_m(x)| \le 1 & |x| \le 1. \end{cases} \tag{8.23}$$

Therefore, we obtain

$$\begin{aligned} \frac{\langle\Psi|\mathcal{O}(\mathbb{1}-G)\mathcal{O}'|\Psi\rangle}{\|\mathcal{O}^\dagger|\Psi\rangle\|\|\mathcal{O}'|\Psi\rangle\|} &\le \max_{0\le x\le 1-\delta} |Q_m(x)| = \max_{0\le x\le 1-\delta} \frac{|T_m(\frac{2x}{1-\delta}-1)|}{T_m(\frac{2}{1-\delta}-1)} \\ &\le 1\cdot\left(\frac{1}{2}\mathrm{e}^{2m\sqrt{\delta}}\right)^{-1} \\ &= 2\exp\left(-2m\sqrt{\frac{\epsilon}{g^2+\epsilon}}\right) \\ &= 2\exp\left(-\frac{2(\tilde{d}(\mathcal{O},\mathcal{O}')-2)}{2c-1}\sqrt{\frac{\epsilon}{g^2+\epsilon}}\right). \quad \square \end{aligned} \tag{8.24}$$